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MATHEMATICS
TARGET IIT JEE

TRIGONOMETRIC RATIOS & IDENTITIES (PHASE–I)

THEORY AND EXERCISE BOOKLET

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JEE Syllabus :

Trigonometric functions, their periodicity and graphs, addition and subtraction formulae, formulae involving multiple and sub-multiple angles

A. BASIC TRIGONOMETRIC IDENTITIES

(a) $\sin^2\theta + \cos^2\theta = 1$; $-1 \le \sin\theta \le 1$; $-1 \le \cos\theta \le 1 \ \forall\ \theta \in R$

(b) $\sec^2\theta - \tan^2\theta = 1$; $|\sec\theta| \ge 1 \ \forall\ \theta \in R$

(c) $\text{cosec}^2\theta - \cot^2\theta = 1$; $|\text{cosec}\,\theta| \ge 1 \ \forall\ \theta \in R$

Important Trigonometric Ratios :

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(b) $\sin\frac{(2n+1)\pi}{2} = (-1)^n$ & $\cos\frac{(2n+1)\pi}{2} = 0$ where $n \in I$

Trigonometric Functions Of Allied Angles :

If θ is any angle , then $-\theta$, $90 \pm \theta$, $180 \pm \theta$, $270 \pm \theta$, $360 \pm \theta$ etc. are called **ALLIED ANGLES** .

(a) $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$; $\tan(-\theta) = -\tan\theta$

(b) $\sin(90°-\theta) = \cos\theta$; $\cos(90°-\theta) = \sin\theta$; $\tan(90°-\theta) = \cot\theta$

(c) $\sin(90°+\theta) = \cos\theta$; $\cos(90°+\theta) = -\sin\theta$; $\tan(90°+\theta) = -\cot\theta$

(d) $\sin(180°-\theta) = \sin\theta$; $\cos(180°-\theta) = -\cos\theta$; $\tan(180°-\theta) = -\tan\theta$

(e) $\sin(180°+\theta) = -\sin\theta$; $\cos(180°+\theta) = -\cos\theta$; $\tan(180°+\theta) = \tan\theta$

(f) $\sin(270°-\theta) = -\cos\theta$; $\cos(270°-\theta) = -\sin\theta$; $\tan(270°-\theta) = \cot\theta$

(g) $\sin(270°+\theta) = -\cos\theta$; $\cos(270°+\theta) = \sin\theta$; $\tan(270°+\theta) = -\cot\theta$

Ex.1 Express 1·2 radians in degree measure.

Sol. 1·2 radians $= 1{\cdot}2 \times \frac{180}{\pi}$ degrees $= 1{\cdot}2 \times \frac{180}{22/7}$ $\left[\because \pi = \frac{22}{7} \text{ (approx).}\right]$

$= \frac{1{\cdot}2 \times 180 \times 7}{22} = 68{\cdot}7272 = 68^o\,({\cdot}7272 \times 60)' = 68^o\,(43{\cdot}63)'$

$= 68^o\,43'\,({\cdot}63 \times 60)'' = 68^o\,43'\,37{\cdot}8''$

Ex.2 Calculate $\sin\alpha$ if $\cos\alpha = -\frac{9}{11}$ and $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$.

Sol. For any angle α belonging to the indicated interval $\sin\alpha$ is negative, and therefore $\sin\alpha = -\sqrt{1-\cos^2\alpha}$

$= -\sqrt{1-\left(-\frac{9}{11}\right)^2} = -\frac{2\sqrt{10}}{11}$.

Ex.3 Calculate $\tan\alpha$ if $\cos\alpha = -\frac{\sqrt{5}}{5}$ and $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$.

Sol. For any angle α belonging to the indicated interval $\tan\alpha$ is positive and $\cos\alpha$ is negative, and therefore $\tan\alpha = \frac{-\sqrt{1-\cos^2\alpha}}{\cos\alpha} = 2$.

Ex.4 Given that $5\cos^2\alpha - 2\sin\alpha - 2 = 0$ $\left(\frac{5\pi}{4} < \alpha < \frac{7\pi}{4}\right)$, then find the value of $\cot\frac{\alpha}{2}$.

Sol. Making a quadratic equation in $\sin^2\alpha$

$(\sin\alpha + 1)(5\sin\alpha - 3) = 0$ $\sin\alpha = -1$ $\sin\alpha = \frac{3}{5}$ not possible as $\left(\frac{5\pi}{4} < \alpha < \frac{7\pi}{4}\right)$

$\alpha = \frac{3\pi}{2}, \frac{\alpha}{2} = \frac{3\pi}{4} \Rightarrow \cot\frac{3\pi}{4} = 1$

Ex.5 Prove that $3(\sin x - \cos x)^4 + 4(\sin^6 x + \cos^6 x) + 6(\sin x + \cos x)^2 = 13$

Sol. L.H.S. $= 3[(\sin x - \cos x)^2]^2 + 4[(\sin^2 x)^3 + (\cos^2 x)^3)]$
$+ 6(\sin^2 x + \cos^2 x + 2\sin x\cos x)$
$= 3(\sin^2 x + \cos^2 x - 2\sin x\cos x)^2 + 4(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x\cos^2 x)]$
$+ 6(\sin^2 x + \cos^2 x + 2\sin x\cos x)$
$= 3(1 - 2\sin x\cos x)^2 + 4[(\sin^4 x + \cos^4 x) - \sin^2 x\cos^2 x] + 6(1 + 2\sin x . \cos x)$
$= 3(1 + 4\sin^2 x\cos^2 x - 4\sin x\cos x) + 4[(\sin^2 x + \cos^2 x)^2$
$- 2\sin^2 x\cos^2 x - \sin^2 x\cos^2 x] + 6 + 12\sin x\cos x$
$= 3 + 12\sin^2 x\cos^2 x - 12\sin x\cos x + 4(1 - 3\sin^2 x\cos^2 x) + 6 + 12\sin x\cos x$
$= 3 + 12\sin^2 x\cos^2 x + 4 - 12\sin^2 x\cos^2 x + 6 = 13$

Ex.6 If $\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1$, Prove that $\sin^4 A + \sin^4 B = 2\sin^2 A\sin^2 B$

Sol. Given, $\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1 = \cos^2 A + \sin^2 A$

or, $\frac{\cos^4 A}{\cos^2 B} - \cos^2 A = \sin^2 A - \frac{\sin^4 A}{\sin^2 B}$

or, $\frac{\cos^2 A(\cos^2 A - \cos^2 B)}{\cos^2 B} = \frac{\sin^2 A(\sin^2 B - \sin^2 A)}{\sin^2 B}$

or, $\frac{\cos^2 A}{\cos^2 B}(\cos^2 A - \cos^2 B) = \frac{\sin^2 A}{\sin^2 B}[(1 - \cos^2 B) - (1 - \cos^2 A)]$

or, $(\cos^2 A - \cos^2 B)\left(\frac{\cos^2 A}{\cos^2 B} - \frac{\sin^2 A}{\sin^2 B}\right) = 0$

When $\cos^2 A - \cos^2 B = 0$, $\cos^2 A = \cos^2 B$

when $\frac{\cos^2 A}{\cos^2 B} - \frac{\sin^2 A}{\sin^2 B} = 0$, $\cos^2 A\sin^2 B = \sin^2 A\cos^2 B$

or, $\cos^2 A\,(1 - \cos^2 B) = (1 - \cos^2 A)\cos^2 B$
or, $\cos^2 A - \cos^2 A \cos^2 B = \cos^2 B - \cos^2 A \cos^2 B$
or, $\cos^2 A = \cos^2 B$

Thus $\left.\begin{array}{r}\cos^2 A = \cos^2 B \\ \therefore 1-\sin^2 A = 1-\sin^2 B \text{ or}, \sin^2 A = \sin^2 B\end{array}\right\}$

L.H.S. = $\sin^4 A + \sin^4 B = (\sin^2 A - \sin^2 B)^2 + 2\sin^2 A \sin^2 B$
$= 2\sin^2 A \sin^2 B$ = R.H.S. $[\because \sin^2 A = \sin^2 B]$

Ex.7 Simplify the expression $\dfrac{1}{\sqrt{b-a}}\cdot\dfrac{\sqrt{\dfrac{b-a}{a}}\sin x}{\sqrt{1+\left(\sqrt{\dfrac{b-a}{a}}\sin x\right)^2}}\sqrt{a+b\tan^2 x}$ where $b > a > 0$.

Sol. After a few simple manipulations, this expression (for brevity denote it by P) can be rewritten

$$P = \frac{\sin x\sqrt{a+b\tan^2 x}}{\sqrt{a+(b-a)\sin^2 x}} = \frac{\sin x\sqrt{a+b\tan^2 x}}{\sqrt{a\cos^2 x + b\sin^2 x}}$$

Some students handle this as follows:

$$\sqrt{a+b\tan^2 x} = \sqrt{a+b\frac{\sin^2 x}{\cos^2 x}} = \frac{\sqrt{a\cos^2 x + b\sin^2 x}}{\cos x}$$

and get a wrong answer: P = tan x. In this transformation what we actually have to simplify is the expression $\sqrt{\cos^2 x}$ which is equal to |cos x|. And so the final result is P = sinx / |cos x|.

Ex.8 If $\tan\theta = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots\infty}}}$ where $\theta \in (0, 2\pi)$, find the possible values of θ.

Sol. Let $\tan\theta = x = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots\infty}}} = \dfrac{1}{2+x}$

$x^2 + 2x - 1 = 0 \quad \Rightarrow \quad x = \dfrac{-2\pm\sqrt{8}}{2} = (\sqrt{2}-1) \quad \because \quad -\sqrt{2}-1$ is not b/w $(0, 2\pi)$

$\therefore \quad \tan\theta = \sqrt{2}-1 \quad \Rightarrow \quad \theta = \dfrac{\pi}{8}$ or $\dfrac{9\pi}{8}$

Ex.9 Simplify $\dfrac{\cos^3\left(\dfrac{\pi}{2}+\theta\right)\cot(3\pi+\theta)\sec(\theta-3\pi)\operatorname{cosec}\left(\dfrac{3\pi}{2}-\theta\right)}{\tan^2(\theta-\pi)\sin(\theta-2\pi)}$.

Sol. $\dfrac{(-\sin^3\theta)(\cot\theta)(-\sec\theta)(-\sec\theta)}{\tan^2\theta\sin\theta} = -\dfrac{\sin^3\theta\cdot\cos^2\theta\cdot\cos\theta}{\sin^2\theta\cdot\sin\theta\cdot\cos^2\theta\cdot\sin\theta} = -\dfrac{\cos\theta}{\sin\theta} = -\cot\theta$

Ex.10 If the expression

$\cos\left(x-\dfrac{3\pi}{2}\right) + \sin\left(\dfrac{3\pi}{2}+x\right) + \sin(32\pi + x) - 18\cos(19\pi - x) + \cos(56\pi + x) - 9\sin(x + 17\pi)$

is expressed in the form of a sin x + b cos x find the value of a + b.

Sol. – sin x – cos x + sin x + 18 cos x + cos x + 9 sin x

18 cos x + 9 sin x = a sin x + b cos x

$\therefore$ a = 9, b = 18 $\qquad\therefore$ a + b = 27 Ans.

Alternatively: put x = 0 and $x = \dfrac{\pi}{2}$ to get a and b directly

Ex.11 If $\dfrac{\sin^4\alpha}{a}+\dfrac{\cos^4\alpha}{b}=\dfrac{1}{a+b}$, prove that $\dfrac{\sin^8\alpha}{a^3}+\dfrac{\cos^8\alpha}{b^3}=\dfrac{1}{(a+b)^3}$

Sol. Given $\dfrac{\sin^4\alpha}{a}+\dfrac{\cos^4\alpha}{b}=\dfrac{1}{a+b}$

or, $b(a + b)\sin^4\alpha + a(a + b)(1 - \sin^2\alpha)^2 = ab$.

or, $b(a + b)\sin^4\alpha + a(a + b)(1 + \sin^4\alpha - 2\sin^2\alpha) = ab$

or, $(a + b)^2\sin^4\alpha - 2a(a + b)\sin^2\alpha + a^2 + ab = ab$

or, $(a + b)^2\sin^4\alpha - 2(a + b)\sin^2\alpha.\ a + a^2 = 0$

or, $[(a + b)\sin^2\alpha - a]^2 = 0$

or, $(a + b)\sin^2\alpha - a = 0$

$\Rightarrow$ $\sin^2\alpha = \dfrac{a}{a+b}$ $\qquad\therefore$ $\cos^2\alpha = \dfrac{b}{a+b}$

Now, $\dfrac{\sin^8\alpha}{a^3}+\dfrac{\cos^8\alpha}{b^3}=\dfrac{a^4}{(a+b)^4.a^3}+\dfrac{b^4}{(a+b)^4b^3}$

$= \dfrac{a}{(a+b)^4}+\dfrac{b}{(a+b)^4}=\dfrac{a+b}{(a+b)^4}=\dfrac{1}{(a+b)^3}$

Ex.12 If $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ and $y = \log_{10}(\tan x + \sec x)$. Then the expression $E = \dfrac{10^y - 10^{-y}}{2}$ simplifies to one of the six trigonometric functions. find the trigonometric function.

Sol. $y = \log_{10}(\tan x + \sec x)$, $y = \log_{10}\left(\dfrac{1+\sin x}{\cos x}\right)$

$$E = \frac{10^y - 10^{-y}}{2} = \frac{\left(\dfrac{1+\sin x}{\cos x}\right) - \left(\dfrac{\cos x}{1+\sin x}\right)}{2} = \frac{1+\sin^2 x + 2\sin x - \cos^2 x}{2\cos x(1+\sin x)}$$

$$= \frac{2\sin^2 x + 2\sin x}{2\cos x(1+\sin x)} = \frac{2\sin x(1+\sin x)}{2\cos x(1+\sin x)} = \tan x$$

B. TRIGONOMETRIC FUNCTIONS OF SUM OR DIFFERENCE OF TWO ANGLES

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(c) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B) \,.\, \sin(A-B)$

(d) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B) \,.\, \cos(A-B)$

(e) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ **(f)** $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

Factorisation Of The Sum Or Difference Of Two sines Or cosines :

(a) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ **(b)** $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ **(d)** $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

Transformation Of Products Into Sum Or Difference Of sines & cosines :

(a) $2\sin A \cos B = \sin(A+B) + \sin(A-B)$ **(b)** $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

(c) $2\cos A \cos B = \cos(A+B) + \cos(A-B)$ **(d)** $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

Ex.13 Suppose x and y are real numbers such that $\tan x + \tan y = 42$ and $\cot x + \cot y = 49$. Find the value of $\tan(x + y)$.

Sol. $\tan x + \tan y = 42$ and $\cot x + \cot y = 49$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

now, $\cot x + \cot y = 49$ $\Rightarrow$ $\dfrac{1}{\tan x} + \dfrac{1}{\tan y} = 49$ $\Rightarrow$ $\dfrac{\tan y + \tan x}{\tan x \cdot \tan y} = 49$

$$\tan x \cdot \tan y = \frac{\tan x + \tan y}{49} = \frac{42}{49} = \frac{6}{7}$$

$$\tan(x+y) = \frac{42}{1-(6/7)} = \frac{42}{1/7} = 294 \text{ Ans.}$$

Ex.14 If x $\sin\theta$ = y $\sin\left(\theta+\frac{2\pi}{3}\right)$ = z $\sin\left(\theta+\frac{4\pi}{3}\right)$ then :

(A) x + y + z = 0 (B) xy + yz + zx = 0 (C) xyz + x + y + z = 1 (D) none

Sol. x sin θ = $y\left(-\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta\right) \Rightarrow \frac{x}{y} = \frac{\sqrt{3}}{2}\cot\theta - \frac{1}{2}$

similarly $\frac{x}{z} = -\frac{\sqrt{3}}{2}\cot\theta - \frac{1}{2}$ ⇒ on adding $\frac{x}{z}+\frac{x}{y} = -1$ ⇒ xy + yz + zx = 0 Ans. B

Ex.15 In any triangle if $\tan\frac{A}{2} = \frac{5}{6}$ and $\tan\frac{B}{2} = \frac{20}{37}$ then find the value of tan C.

Sol. $\tan\left(\frac{A}{2}+\frac{B}{2}\right) = \frac{\frac{5}{6}+\frac{20}{37}}{1-\frac{5\times20}{6\times37}}$; $\cot\frac{C}{2} = \frac{185+120}{222-100} = \frac{305}{122}$; $\tan\frac{C}{2} = \frac{122}{305}$ $\left(\because \tan C = \frac{2\tan C/2}{1-\tan^2 C/2}\right)$

$$\therefore \quad \tan C = \frac{\frac{244}{305}}{1-\left(\frac{122}{305}\right)^2} = \frac{244\times305}{(305-122)(305+122)} = \frac{244\times305}{183\times427} = \frac{4\times5}{3\times7} = \frac{20}{21}$$

Ex.16 Find θ satisfying the equation, tan 15° · tan 25° · tan 35° = tan θ, where θ ∈ (0, 15°).

Sol. LHS = tan 15° · tan (30° – 5°) · tan (30° + 5°)

let t = tan 30° and m = tan 5°

$$\tan\theta = \tan 15° \cdot \frac{t-m}{1+tm} \cdot \frac{t+m}{1-tm} = \tan\left(3(5°)\right) \cdot \frac{t^2-m^2}{1-t^2m^2} = \frac{3m-m^3}{1-3m^2}\cdot\frac{1-3m^2}{3-m^2}$$

$$= \frac{m(3-m^2)}{(1-3m^2)}\cdot\frac{(1-3m^2)}{3-m^2} = m = \tan 5°.\ \text{Hence } \theta = 5° \qquad \left\{\begin{array}{l}\because \tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x};\\ t = \tan 30° \Rightarrow t^2 = 1/3\end{array}\right\}$$

Ex.17 If tan A & tan B are the roots of the quadratic equation, $ax^2 + bx + c = 0$ then evaluate $a\sin^2(A+B) + b\sin(A+B)\,.\,\cos(A+B) + c\cos^2(A+B)$.

Sol. $\tan A + \tan B = -\frac{b}{a}$; $\tan A \,.\, \tan B = \frac{c}{a}$

$$\tan(A+B) = \frac{-\frac{b}{a}}{1-\frac{c}{a}} = \frac{b}{c-a}$$

Now $E = \cos^2 (A + B) [a \tan^2 (A + B) + b \tan (A + B) + c]$

$$= \frac{1}{1+\frac{b^2}{(c-a)^2}}\left[\frac{ab^2}{(c-a)^2}+\frac{b^2}{c-a}+c\right] = \frac{(c-a)^2}{b^2+(c-a)^2}\left[\frac{b^2}{c-a}\left(\frac{a}{c-a}+1\right)+c\right]$$

$$= \frac{(c-a)^2}{b^2+(c-a)^2}\left[\frac{b^2c}{(c-a)^2}+c\right] \qquad E = c$$

Ex.18 The value of the expression, $\frac{\cos^3 2x + 3\cos 2x}{\cos^6 x - \sin^6 x}$ wherever defined is independent of x. Without allotting a particular value of x, find the value of this constant.

Sol. $$\frac{\cos^3 2x+3\cos 2x}{\cos^6 x-\sin^6 x} = \frac{\cos^3 2x+3\cos 2x}{(\cos^2 x)^3-(\sin^2 x)^3} = \frac{\cos^3 2x+3\cos 2x}{\cos^3 2x+3\sin^2 x\cos^2 x(\cos 2x)}$$

$$= \frac{\cos^2 2x+3}{\cos^2 2x+\frac{3}{4}\sin^2 2x} = \frac{4(\cos^2 2x+3)}{4\cos^2 2x+3-3\cos^2 2x} = \frac{4(\cos^2 2x+3)}{(\cos^2 2x+3)} = 4 \text{ Ans.}$$

Ex.19 Show that $\cos^2A + \cos^2(A + B) + 2 \cos A \cos(180° + B) \cdot \cos(360° + A + B)$ is independent of A. Hence find its value when $B = 810°$.

Sol. $\cos^2A + \cos^2(A + B) - [2 \cos A \cdot \cos B \cdot \cos (A + B)]$
$\cos^2A + \cos^2(A + B) - [\{\cos(A + B) + \cos(A - B) \} \cos (A + B)]$
$\cos^2A + \cos^2(A + B) - \cos^2(A + B) - (\cos^2A - \sin^2B)$
$= \sin^2B$ which is independent of A   now, $\sin^2(810°) = \sin^2(720° + 90°) = \sin^2 90° = 1$ Ans.

Ex.20 Simplify: $\cos x \cdot \sin(y - z) + \cos y \cdot \sin(z - x) + \cos z \cdot \sin (x - y)$ where $x, y, z \in R$.

Sol. $(1/2)[\sin(y - z + x) + \sin(y - z - x) + \sin(z - x + y) + \sin(z - x - y) + \sin(x - y + z) + \sin(x - y - z)] = 0$

C. MULTIPLE ANGLES AND SUB-MULTIPLE ANGLES

(a) $\sin 2A = 2 \sin A \cos A$; $\sin\theta = 2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}$

(b) $\cos 2A = \cos^2A - \sin^2A = 2\cos^2A - 1 = 1 - 2 \sin^2A$;

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$.

$2 \cos^2A = 1 + \cos 2A$, $2\sin^2A = 1 - \cos 2A$;

$2 \cos^2\frac{\theta}{2} = 1 + \cos\theta$, $2 \sin^2\frac{\theta}{2} = 1 - \cos\theta$.

(c) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$; $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(d) $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$, $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$ **(e)** $\sin 3A = 3 \sin A - 4 \sin^3A$

(f) $\cos 3A = 4 \cos^3A - 3 \cos A$ **(g)** $\tan 3A = \frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

Important Trigonometric Ratios

(i) $\sin 15°$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ = $\cos 75°$ or $\cos\frac{5\pi}{12}$;

$\cos 15°$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$ = $\sin 75°$ or $\sin\frac{5\pi}{12}$;

$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$ = $\cot 75°$; $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$ = $\cot 15°$

(ii) $\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$; $\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$; $\tan\frac{\pi}{8} = \sqrt{2}-1$; $\tan\frac{3\pi}{8} = \sqrt{2}+1$

(iii) $\sin\frac{\pi}{10}$ or $\sin 18° = \frac{\sqrt{5}-1}{4}$ & $\cos 36°$ or $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

Ex.21 If $\cot\theta = 1/2$, then find the values of $\sin 2\theta$ and $\cos 2\theta$.

Sol. $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\cdot 2}{1+4} = \frac{4}{5}$; $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-4}{1+4} = -\frac{3}{5}$

Ex.22 Prove that $\frac{\tan 8\theta}{\tan\theta} = (1+\sec 2\theta)(1+\sec 4\theta)(1+\sec 8\theta)$.

Sol. RHS $= \frac{1+\cos 2\theta}{\cos 2\theta}\times\frac{1+\cos 4\theta}{\cos 4\theta}\times\frac{1+\cos 8\theta}{\cos 8\theta} = \frac{2\cos^2\theta\times 2\cos^2 2\theta\times 2\cos^2 4\theta}{\cos 2\theta\cos 4\theta\cos 8\theta}$

$= \frac{[8\cos\theta\cos 2\theta\cos 4\theta]\cos\theta}{\cos 8\theta} = \frac{\left[\frac{\sin 8\theta}{\sin\theta}\right]\cos\theta}{\cos 8\theta}$

Ex.23 If $x = 7.5°$ then find the value of $\frac{\cos x-\cos 3x}{\sin 3x-\sin x}$.

Sol. $\frac{\cos x-\cos 3x}{\sin 3x-\sin x} = \frac{2\sin 2x\sin x}{2\sin x\cos 2x} = \tan 2x = \tan(2\times 7.5) = \tan 15° = 2-\sqrt{3}$ Ans.

Ex.24 Prove the identity,

$$\cos\left(\frac{3\pi}{2}+4\alpha\right) + \sin(3\pi-8\alpha) - \sin(4\pi-12\alpha) = 4\cos 2\alpha\cos 4\alpha\sin 6\alpha .$$

Sol. LHS : $\sin 4\alpha + \sin 8\alpha + \sin 12\alpha$

$= 2\sin 8\alpha\cos 4\alpha + \sin 8\alpha = 2\sin 8\alpha\cos 4\alpha + 2\sin 4\alpha\cos 4\alpha$

$= 2\cos 4\alpha[\sin 8\alpha + \sin 4\alpha] = 2\cos 4\alpha[2\sin 6\alpha\cos 2\alpha] = 4\cos 2\alpha\cos 4\alpha\sin 6\alpha$

Ex.25 Calculate 4 sin$\left(1+\frac{\pi}{6}\right)$ cos $\left(1+\frac{\pi}{3}\right)$.

Sol. $4\sin\left(1+\frac{\pi}{6}\right)\cos\left(1+\frac{\pi}{3}\right) = 2\left[\sin\left(1+\frac{\pi}{6}+1+\frac{\pi}{3}\right)+\sin\left(1+\frac{\pi}{6}-1-\frac{\pi}{3}\right)\right]$

$= 2\left[\sin\left(2+\frac{\pi}{2}\right)+\sin\left(-\frac{\pi}{6}\right)\right] = 2\left[\sin\left(\frac{\pi}{2}-(-2)\right)-\frac{1}{2}\right]$ = 2cos (–2) – 1 = 2 cos 2 – 1.

Thus, 4 sin$\left(1+\frac{\pi}{6}\right)$ cos$\left(1+\frac{\pi}{3}\right)$ = 2 cos 2 – 1.

Ex.26 If $\cos\alpha = \frac{2\cos\beta - 1}{2-\cos\beta}$ then find the value of $\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$ ($0 < \alpha < \pi$ and $0 < \beta < \pi$)

Sol. $\frac{1}{\cos\alpha} = \frac{2-\cos\beta}{2\cos\beta - 1} \Rightarrow \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{3(1-\cos\beta)}{1+\cos\beta}$ (Componendo & dividendo)

$\Rightarrow \tan^2\frac{\alpha}{2} = 3\tan^2\frac{\beta}{2} \Rightarrow \tan^2\frac{\alpha}{2}\cot^2\frac{\beta}{2} = 3$ Ans. $\sqrt{3}$

Ex.27 Calculate $\cos\frac{\alpha}{2}$ if $\sin\alpha = \frac{4}{5}$ and $\alpha \in \left(-\frac{3\pi}{2}, -\pi\right)$.

Sol. First of all we seek cos α. Since cos α is negative for any angle α belonging to the indicated interval, we have $\cos\alpha = -\sqrt{1-\sin^2\alpha} = -\frac{3}{5}$.

Since $\alpha \in \left(-\frac{3\pi}{2}, -\pi\right)$, it follows that $\frac{\alpha}{2} \in \left(-\frac{3\pi}{4}, -\frac{\pi}{2}\right)$. For any angle $\frac{\alpha}{2}$ belonging to this interval $\cos\frac{\alpha}{2}$ is also negative, and therefore $\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} = -\frac{\sqrt{5}}{5}$. Thus $\cos\frac{\alpha}{2} = -\frac{\sqrt{5}}{5}$.

Ex.28 Calculate $\sin\frac{\alpha}{2}$ if $\sin\alpha = \frac{4\sqrt{2}}{9}$ and $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$.

Sol. Since cos α is negative for any angle α belonging to the indicated interval, we have $\cos\alpha = -\sqrt{1-\sin^2\alpha} = -\frac{7}{9}$.

Since $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$, it follows that $\frac{\alpha}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$. For any angle $\sin\frac{\alpha}{2}$ is positive, and therefore

$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \frac{2\sqrt{2}}{3}$. Thus $\sin\frac{\alpha}{2} = \frac{2\sqrt{2}}{3}$.

Ex.29 Calculate $\tan\frac{\alpha}{2}$ if $\cos 2\alpha = \frac{7}{32}$ and $\alpha \in \left(-\pi, -\frac{3\pi}{4}\right)$.

Sol. Since $\cos\alpha$ is negative for an angle α belonging to the indicated interval,

we have $\cos\alpha = \sqrt{\frac{1+\cos 2\alpha}{2}} = -\frac{\sqrt{39}}{8}$.

Since $\alpha \in \left(-\pi, -\frac{3\pi}{4}\right)$, it follows that $\frac{\alpha}{2} \in \left(-\frac{\pi}{2}, -\frac{3\pi}{8}\right)$. For any angle $\frac{\alpha}{2}$ belonging to this interval $\tan\frac{\alpha}{2}$

is negative, and therefore $\tan\frac{\alpha}{2} - \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \Rightarrow \tan\frac{\alpha}{2} = -\frac{8+\sqrt{39}}{5}$.

Ex.30 If sin A = 12/13. Find the value of tan A/2.

Sol. sinA = 12/13 $\Rightarrow$ A is in I quadrant or II quadrant

$\therefore$ cos A can be 5/13 or – 5/13

Case-I: A is in I quadrant

$\therefore$ $0 < A/2 < \pi/4$ $\therefore$ cos A = 5/13

$$\therefore \quad \tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}} = \sqrt{\frac{1-(5/13)}{1+(5/13)}} = \sqrt{\frac{8}{18}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$$

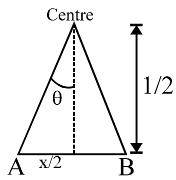
Case-II: A is in II quadrant

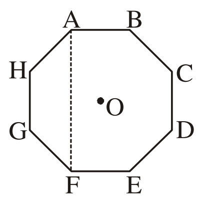
$\therefore$ $\pi/4 < A/2 < \pi/2$ $\therefore$ cos A = – 5/13

$$\therefore \quad \tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}} = \sqrt{\frac{1+(5/13)}{1-(5/13)}} = \sqrt{\frac{18}{8}} = \sqrt{\frac{9}{4}} = \frac{3}{2} \quad \therefore \tan\frac{A}{2} \text{ can be } \frac{2}{3} \text{ or } \frac{3}{2}$$

Ex.31 The figure (not drawn to scale) shows a regular octagon ABCDEFGH with diagonal AF = 1. Find the numerical value of the side of the octagon.

Sol. $\theta = 22.5°$ ($\angle$ AOB = 45º)





$\tan 22.5° = \frac{x}{2}\cdot\frac{2}{1}$

$x = \tan 22.5° = \sqrt{2}-1$

Ex.32 If $\frac{\tan\theta}{\tan\theta - \tan 3\theta} = \frac{1}{3}$, find the value of $\frac{\cot\theta}{\cot\theta - \cot 3\theta}$.

Sol. $\frac{\tan\theta}{\tan\theta - \tan 3\theta} = \frac{1}{3}$ $\Rightarrow$ $3\tan\theta = \tan\theta - \tan 3\theta$ $\Rightarrow$ $2\tan\theta + \tan 3\theta = 0$

$2\tan\theta + \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = 0$, $\quad 2(1 - 3\tan^2\theta) + 3 - \tan^2\theta = 0 \quad \Rightarrow \quad \tan^2\theta = \frac{5}{7}$

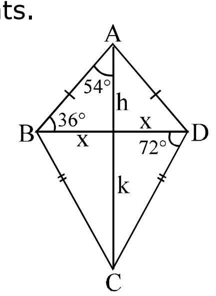
now, $\dfrac{\cot\theta}{\cot\theta-\cot 3\theta}=\dfrac{\tan 3\theta}{\tan 3\theta-\tan\theta}=\dfrac{3\tan\theta-\tan^3\theta}{(1-3\tan^2\theta)\left(\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}-\tan\theta\right)}$

$=\dfrac{\tan\theta(3-\tan^2\theta)(1-3\tan^2\theta)}{\tan\theta(1-3\tan^2\theta)(3-\tan^2\theta-1+3\tan^2\theta)}\quad=\dfrac{3-\tan^2\theta}{2(1+\tan^2\theta)}=\dfrac{3-(5/7)}{2(1+(5/7))}=\dfrac{16}{2\cdot 12}=\dfrac{2}{3}$ Ans.

Alternatively: Prove that $\dfrac{\tan\theta}{\tan\theta-\tan 3\theta}+\dfrac{\cot\theta}{\cot\theta-\cot 3\theta}=1$ now proceed

Ex.33 In a kite ABCD, AB = AD and CB = CD. If $\angle A = 108°$ and $\angle C = 36°$ then the ratio of the area of ΔABD to the area of ΔCBD can be written in the form $\dfrac{a-b\tan^2 36°}{c}$ where a, b and c are relatively prime positive integers. Determine the ordered triple (a, b, c).

Sol. Since the triangles ABD and CBD have a common base, hence the ratio of their areas equals the ratio of their heights.



Since $\tan 36° = \dfrac{h}{x}$, then $h = x\tan 36°$.

|||ly $\tan 72° = \dfrac{k}{x}$ then $k = x\tan 72°$.

Hence, $\dfrac{h}{k}=\dfrac{x\tan 36°}{x\tan 72°}=\dfrac{\tan 36°}{\dfrac{2\tan 36°}{1-\tan^2 36°}}=\dfrac{1-\tan^2 36°}{2}$

Then ordered triple (a, b, c) is (1,1, 2) Ans.

Ex.34 If α, β, γ and δ be the roots of the equation, $2\cos 2\theta - 2\cos\theta + 1 = 0$, all lying in the interval $[0, 2\pi]$ then find the value of the product, $\cos\alpha \,.\, \cos\beta \,.\, \cos\gamma \,.\, \cos\delta$.

Sol. $4\cos^2\theta - 2\cos\theta - 1 = 0$ $\qquad \cos\theta = \dfrac{2\pm\sqrt{4+16}}{8} = \dfrac{1\pm\sqrt{5}}{4}$

$\cos\theta = \dfrac{\sqrt{5}+1}{4}$ or $\cos\theta = -\dfrac{\sqrt{5}-1}{4} = -\sin\dfrac{\pi}{10} = \cos\left(\dfrac{5\pi}{10}+\dfrac{\pi}{10}\right) = \cos\dfrac{6\pi}{10}$

$\Rightarrow \theta = \dfrac{\pi}{5}$ or $\dfrac{9\pi}{5}$; $\dfrac{3\pi}{5}$ or $\dfrac{7\pi}{5}$

Hence $P = \cos\dfrac{\pi}{5}\cos\dfrac{3\pi}{5}\cos\dfrac{7\pi}{5}\cos\dfrac{9\pi}{5} = \dfrac{1}{16}$

Ex.35 Find the positive integers p, q, r, s satisfying $\tan\frac{\pi}{24} = \left(\sqrt{p}-\sqrt{q}\right)\left(\sqrt{r}-s\right)$.

Sol. Solving using the Identity $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$ where $\theta = \frac{\pi}{12}$

$$\tan\frac{\pi}{24} = \frac{1-\cos 15°}{\sin 15°} = \frac{1-\frac{\sqrt{6}+\sqrt{2}}{4}}{\frac{\sqrt{6}-\sqrt{2}}{4}} = \frac{4-\left(\sqrt{6}+\sqrt{2}\right)}{\sqrt{6}-\sqrt{2}} = \frac{[4-(\sqrt{6}+\sqrt{2})][\sqrt{6}+\sqrt{2}]}{4}$$

$$= \frac{4(\sqrt{6}+\sqrt{2})-(8+4\sqrt{3})}{4} = (\sqrt{6}+\sqrt{2}) - (2+\sqrt{3})$$

$$= \left(\sqrt{6}-\sqrt{3}\right)-\left(2-\sqrt{2}\right) = \sqrt{3}(\sqrt{2}-1)-\sqrt{2}(\sqrt{2}-1) = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$$

hence p = 3, q = 2; r = 2; s = 1

Ex.36 If $\tan\left(\frac{\pi}{4}+\frac{y}{2}\right) = \tan^3\left(\frac{\pi}{4}+\frac{x}{2}\right)$ then prove that $\sin y = \frac{\sin x(3+\sin^2 x)}{1+3\sin^2 x}$.

Sol. $\tan\left(\frac{\pi}{4}+\frac{y}{2}\right) = \tan^3\left(\frac{\pi}{4}+\frac{x}{2}\right)$

$$\frac{\cos(y/2)+\sin(y/2)}{\cos(y/2)-\sin(y/2)} = \left(\frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)}\right)^3$$

squaring both sides

$$\frac{1+\sin y}{1-\sin y} = \left(\frac{1+\sin x}{1-\sin x}\right)^3; \qquad \frac{1+\sin y}{1-\sin y} = \frac{1+\sin^3 x+3\sin x+3\sin^2 x}{1-\sin^3 x-3\sin x+3\sin^2 x}$$

using C & D

$$\frac{1+\sin y-1+\sin y}{1+\sin y+1-\sin y} = \frac{2\sin^3 x+6\sin x}{2+6\sin^2 x} \quad \therefore \sin y = \frac{\sin x(\sin^2 x+3)}{1+3\sin^2 x}$$ hence proved.

Ex.37 If sin x, $\sin^2 2x$ and cos x · sin 4x form an increasing geometric sequence, find the numerical value of cos 2x. Also find the common ratio of geometric sequence.

Sol. Given sin x, $\sin^2 2x$ and cos x · sin 4x are in G.P. (r > 1 as G.P. is increasing)

$\Rightarrow$ $\sin^4 2x = (\sin x)(\cos x)(\sin 4x)$ $\qquad\Rightarrow$ $16\sin^4 x\cos^4 x = \sin x\cos x\sin 4x$

$\Rightarrow$ $16\sin^3 x\cos^3 x = \sin 4x$ $\qquad (\sin x \neq 0, \cos x \neq 0)$

$\Rightarrow$ $16(\sin x\cos x)^3 = 2\sin 2x \cdot \cos 2x$ $\qquad\Rightarrow$ $(\sin 2x)^3 = \sin 2x \cdot \cos 2x$

$\therefore$ $\sin^2 2x = \cos 2x$ $(\sin 2x \neq 0)$, $\qquad 1-\cos^2 2x = \cos 2x$, $\quad y^2+y-1=0$

$\cos 2x = \frac{-1\pm\sqrt{5}}{2}$; cos 2x cannot be $\frac{-\sqrt{5}-1}{2}$ hence rejected $\quad\therefore\quad \cos 2x = \frac{-1+\sqrt{5}}{2}$

$$\sin x = \sqrt{\frac{1-\cos 2x}{2}} = \sqrt{\frac{1-\frac{\sqrt{5}-1}{2}}{2}} = \sqrt{\frac{3-\sqrt{5}}{2}} = \frac{\sqrt{5}-1}{2\sqrt{2}}$$

$\therefore$ $\cos 2x = \dfrac{\sqrt{5}-1}{2}$ $r = \dfrac{\sin^2 2x}{\sin x} = 4 \sin x \cos^2 x = 2 \sin x(1 + \cos 2x)$

$r = \dfrac{\sqrt{5}-1}{\sqrt{2}} \cdot \dfrac{\sqrt{5}+1}{2} = \dfrac{4}{2\sqrt{2}} = \sqrt{2}$ Ans.

Ex.38 Prove using induction or otherwise that, $2\cos\dfrac{\theta}{2^n} = \sqrt{2+\sqrt{2+\sqrt{2+\ldots\ldots\sqrt{2+2\cos\theta}}}}$

where R. H. S. contains n radical signs and $\theta \in (0, \pi)$.

Sol. $2\cos\dfrac{\theta}{2} = \sqrt{2(1+\cos\theta)}$

$2\cos\dfrac{\theta}{2^2} = \sqrt{2\left(1+\cos\frac{\theta}{2}\right)} = \sqrt{2+\sqrt{2(1+\cos\theta)}}$

$2\cos\dfrac{\theta}{2^3} = \sqrt{2\left(1+\cos\frac{\theta}{2^2}\right)} = \sqrt{2+2\cos\frac{\theta}{2^2}} = \sqrt{2+\sqrt{2+\sqrt{2(1+\cos\theta)}}}$ and so on.

In the same way $2\cos\dfrac{\theta}{2^n} = \sqrt{2+\sqrt{2+\sqrt{2+\ldots\ldots+\sqrt{2+2\cos\theta}}}}$

Similarly $2\sin\dfrac{\theta}{2^n} = 2\sqrt{\dfrac{1-\cos\frac{\theta}{2^{n-1}}}{2}} = \sqrt{2-2\cos\dfrac{\theta}{2^{n-1}}}$

$= \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\ldots\ldots\sqrt{2+2\cos\theta}}}}}$ where R. H. S. contains n radical signs

Ex.39 Prove that $\sin^4\dfrac{\pi}{8}+\sin^4\dfrac{3\pi}{8}+\sin^4\dfrac{5\pi}{8}+\sin^4\dfrac{7\pi}{8} = \cos^4\dfrac{\pi}{8}+\cos^4\dfrac{3\pi}{8}+\cos^4\dfrac{5\pi}{8}+\cos^4\dfrac{7\pi}{8}$.

Also find their exact numerical value.

Sol. LHS = $\sin^4\dfrac{\pi}{8}+\sin^4\dfrac{3\pi}{8}+\sin^4\dfrac{5\pi}{8}+\sin^4\dfrac{7\pi}{8}$

$2\left(\sin^4\dfrac{\pi}{8}+\sin^4\dfrac{3\pi}{8}\right) = 2\left(\sin^4\dfrac{\pi}{8}+\cos^4\dfrac{\pi}{8}\right) = 2\left(1-2\sin^2\dfrac{\pi}{8}\cos^2\dfrac{\pi}{8}\right)$

$= 2\left(1-\dfrac{1}{2}\sin^2\dfrac{\pi}{4}\right) = 2\left(1-\dfrac{1}{4}\right) = 2 \times \dfrac{3}{4} = \dfrac{3}{2}$

RHS = $\cos^4\dfrac{\pi}{8}+\cos^4\dfrac{3\pi}{8}+\cos^4\dfrac{5\pi}{8}+\cos^4\dfrac{7\pi}{8}$

$2\left(\cos^4\dfrac{\pi}{8}+\cos^4\dfrac{3\pi}{8}\right) = 2\left(\sin^4\dfrac{\pi}{8}+\cos^4\dfrac{\pi}{8}\right) = 2\left(1-2\sin^2\dfrac{\pi}{8}\cos^2\dfrac{\pi}{8}\right) = 2\left(1-\dfrac{1}{4}\right) = \dfrac{3}{2}$

Ex.40 Show that $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\left(\frac{1}{2}\right)^7$.

Sol. We have $\sin\frac{2\pi}{15} = 2\sin\frac{\pi}{15}\cos\frac{\pi}{15}$, $\sin\frac{4\pi}{15} = 2\sin\frac{2\pi}{15}\cos\frac{2\pi}{15}$,

$\sin\frac{8\pi}{15} = 2\sin\frac{4\pi}{15}\cos\frac{4\pi}{15}$, $\sin\frac{16\pi}{15} = 2\sin\frac{8\pi}{15}\cos\frac{8\pi}{15}$.

Multiplying the equalities and noting that $\sin\frac{16\pi}{15} = -\sin\frac{\pi}{15}$, $\cos\frac{8\pi}{15} = -\cos\frac{7\pi}{15}$.

$\therefore \quad \cos\frac{\pi}{15}\,.\,\cos\frac{2\pi}{15}\,.\,\cos\frac{4\pi}{15}\,.\,\cos\frac{7\pi}{15} = \frac{1}{2^4}$

Further $\quad \cos\frac{5\pi}{15}=\frac{1}{2}$.

and $\sin\frac{6\pi}{15} = 2\sin\frac{3\pi}{15}\cos\frac{3\pi}{15}$, $\sin\frac{12\pi}{15} = 2\sin\frac{6\pi}{15}\cos\frac{6\pi}{15}$.

Hence $\quad \cos\frac{3\pi}{15}\,.\,\cos\frac{6\pi}{15} = \frac{1}{2^2}$.

The rest is obvious.

Ex.41 Prove the following identities

$$\frac{\sin(n+1)x}{\sin nx} = 2\cos x - \cfrac{1}{2\cos x - \cfrac{1}{2\cos x - \ddots - \cfrac{1}{2\cos x}}}$$

(a total of n links);

Sol. Put $2\cos x - \cfrac{1}{2\cos x - \cfrac{1}{2\cos x - \ddots - \cfrac{1}{2\cos x}}} = \frac{P_n}{Q_n}$.

We have $\frac{P_1}{Q_1} = 2\cos x$.

Therefore we may put $\quad P_1 = \frac{\sin 2x}{\sin x}, \quad Q_1 = \frac{\sin x}{\sin x}$.

Further $\quad \frac{P_2}{Q_2} = 2\cos x - \frac{1}{2\cos x} = \frac{4\cos^2 x - 1}{2\cos x}$.

Consequently, we may take $\quad P_2 = \frac{\sin 3x}{\sin x}, \quad Q_2 = \frac{\sin 2x}{\sin x}$.

Let us prove that then $P_n\ \frac{\sin(n+1)x}{\sin x}$, $Q_n = \frac{\sin nx}{\sin x}$ for any n.

Assuming that these formulas are valid for subscripts not exceeding n, let us prove that they also take place at n + 1. We have

$$P_{n+1} = 2\cos x\frac{\sin(n+1)x}{\sin x} - \frac{\sin nx}{\sin x} = \frac{1}{\sin x}\sin(n+2)x.$$

In the same way we find that $Q_{n+1} = \frac{\sin(n+1)x}{\sin x}$, and therefore $\frac{P_n}{Q_n} = \frac{\sin(n+1)x}{\sin x}$

for any whole positive n.

D. CONDITIONAL IDENTITIES

$$\tan(A+B+C) = \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

If A+B+C = π then

(a) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(b) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(c) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

(d) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(e) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

(f) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Ex.42 If A + B + C = π, prove that $\sum\frac{\tan A}{\tan B.\tan C} = \sum\tan A - 2\sum\cot A$.

Sol. $$\text{L H S} = \frac{\tan^2 A+\tan^2 B+\tan^2 C}{\tan A\tan B\tan C} = \frac{(\tan A+\tan B+\tan C)^2-2\Sigma\tan A\tan B}{\Sigma\tan A}$$

$[\because\ \Sigma\tan A = \Pi\tan A]$

$$= \sum\tan A - 2\left[\frac{\tan A\tan B+\tan B\tan C+\tan C\tan A}{\tan A\tan B\tan C}\right] = \sum\tan A - 2\sum\cot A]$$

Ex.43 If A + B + C = π and $\cot\theta = \cot A + \cot B + \cot C$, show that ,
$\sin(A-\theta) . \sin(B-\theta) . \sin(C-\theta) = \sin^3\theta$.

Sol. Given $\cot\theta = \cot A + \cot B + \cot C$ or $\cot\theta - \cot A = \cot B + \cot C$

or $\frac{\sin(A-\theta)}{\sin\theta\ \sin A} = \frac{\sin(B+C)}{\sin B\ \sin C} = \frac{\sin A}{\sin B\ \sin C}$ or $\sin(A-\theta) = \frac{\sin^2 A}{\sin B\ \sin C}\sin\theta$ ....(1)

similarly $\sin(B-\theta) = \frac{\sin^2 B}{\sin C\ \sin B}\sin\theta$ ....(2) $\sin(C-\theta) = \frac{\sin^2 C}{\sin A\ \sin B}\sin\theta$ ....(3)

Multiplying (1) , (2) and (3) we get the result

Ex.44 Find whether a triangle ABC can exists with the tangents of its interior angle satisfying, tan A = x, tan B = x + 1 and tan C = 1 – x for some real value of x. Justify your assertion with adequate reasoning.

Sol. In a triangle $\sum \tan A = \prod \tan A$ (to be proved)

$x + x + 1 + 1 - x = x(1 + x)(1 - x)$

$2 + x = x - x^3$; $x^3 = -2$; $x = -2^{1/3}$

Hence $\tan A = x < 0$ and $\tan B = x + 1 = 1 - 2^{1/3} < 0$

Hence A and B both are obtuse. Which is not possible in a triangle. Hence no such triangle can exist.

Ex.45 Prove that

(a) $\sin^3 A \cos(B - C) + \sin^3 B \cos(C - A) + \sin^3 C \cos(A - B) = 3 \sin A \sin B \sin C$;

(b) $\sin^3 A \sin(B - C) + \sin^3 B \sin(C - A) + \sin^3 C \sin(A - B) = 0$

if $A + B + C = \pi$.

Sol. (a) We have

$\sum \sin^3 A \cos(B - C) = \sum \sin^2 A \sin A \cos(B - C) =$

$= \frac{1}{2} \sum \sin^2 A \{\sin(A + B - C) + \sin(A - B + C)\}$.

But since $A + B + C = \pi$, we have

$\sum \sin^3 A \cos(B - C) = \frac{1}{2} \sum \sin^2 A (\sin 2C + \sin 2B)$

$= \sum \sin^2 A (\sin B \cos B + \sin C \cos C) =$

$= \sin^2 A \sin B \cos B + \sin^2 A \sin C \cos C + \sin^2 B \sin C \cos C + \sin^2 B \sin A \cos A + \sin^2 C \sin A \cos A + \sin^2 C \sin B \cos B$

$= \sin A \sin B (\sin A \cos B + \cos A \sin B)$

$+ \sin A \sin C (\sin A \cos C + \cos A \sin C) + \sin B \sin C (\sin B \cos C + \cos B \sin C)$

$= \sin A \sin B \sin(A + B) + \sin A \sin C \sin(A + C)$

$+ \sin B \sin C \sin(B + C) = 3 \sin A \sin B \sin C$.

(b) We have

$\sum \sin^3 A \sin(B - C) = \sum \sin^2 A \sin A \sin(B - C) = \sum \sin^2 A \sin(B + C) \sin(B - C)$

$= \frac{1}{2} \sum \sin^2 A \{\cos 2C - \cos 2B) = \sum \sin^2 A(\sin^2 B - \sin^2 C)$

$= \sin^2 A \sin^2 B \sin^2 C \sum \left(\frac{1}{\sin^2 C} - \frac{1}{\sin^2 B}\right) = \sin^2 A \sin^2 B \sin^2 C$

$\times \left\{\frac{1}{\sin^2 C} - \frac{1}{\sin^2 B} + \frac{1}{\sin^2 A} - \frac{1}{\sin^2 C} + \frac{1}{\sin^2 B} - \frac{1}{\sin^2 A}\right\} = 0$

Ex.46 Prove the identities

(a) $\sin 3A \sin^3(B - C) + \sin 3B \sin^3(C - A) + \sin 3C \sin^3(A - B) = 0$;

(b) $\sin 3A \cos^3(B - C) + \sin 3B \cos^3(C - A) + \sin 3C \cos^3(A - B) = \sin 3A \sin 3B \sin 3C$

if $A + B + C = \pi$.

Sol.(a) We have $\sin 3x = 3 \sin x - 4 \sin^3 x$.

Therefore $\sum \sin 3A \sin^3(B - C) = \frac{1}{4} \sum \sin 3A \{3 \sin(B - C) - \sin 3(B - C)\}$

$= \frac{3}{4}\sum$ sin 3 (B + C) sin (B – C) – $\frac{1}{4}\sum$ sin 3 (B + C) sin 3 (B – C)

$= \frac{3}{8}\sum$ {cos (2B + 4C) – cos (4B + 2C) – $\frac{1}{8}\sum$ (cos 6C – cos 6B)

$= \frac{3}{8}\sum$ {cos 2 (B + 2C) – cos 2 (C + 2B) + cos 2 (C + 2A)

– cos 2 (A + 2C) + cos 2 (A + 2B) – cos 2 (B + 2A)}

$-\frac{1}{8}$ {cos 6C – cos 6B + cos 6A – cos 6C + cos 6B –cos 6A}.

But cos (2B + 4C) = cos (2B + 4A), cos (2C + 4B) = cos (2C + 4A),
cos (2A + 4C) = cos (2A + 4B).

And so, we finally have $\sum$ sin 3A $\sin^3$ (B – C) = 0.

(b) Since cos 3x = $4\cos^3$ x – 3 cos x, we have $\sum$ sin3A $\cos^3$ (B – C)

$= \frac{1}{4}\sum$ sin 3 (B + C) {cos 3 (B – C) + 3 cos (B – C)}

$= \frac{1}{4}\sum$ sin 3 (B + C) cos 3 (B – C) + $\frac{3}{4}\sum$ sin 3 (B + C) cos (B – C)

$= \frac{1}{8}\sum$ (sin 6B + sin 6C) + $\frac{3}{8}\sum$ {sin (4B + 2C) + sin (2B + 4C)}

$= \frac{1}{4}\sum$ (sin 6A + sin 6B + sin 6C) = sin 3A sin 3B sin 3C.

Ex.47 Given the product p of sines of the angles of a triangle & product q of their cosines, find the cubic equation, whose coefficients are functions of p & q & whose roots are the tangents of the angles of the triangle.

Sol. Given sinA sinB sinC = p ; cosA cosB cosC = q
Hence tanA tanB tanC = tanA + tanB + tanC = p/q
Hence equation of cubic is

$$x^3 - \frac{p}{q}x^2 + \Sigma\tan A \tan Bx - \frac{p}{q} = 0 \qquad ...(i)$$

now $\sum \tan A \tan B = \dfrac{\sin A \sin B \cos C + \sin B \sin C \cos A + \sin C \sin A \cos B}{\cos A \cos B \cos C}$

We know that A + B + C = π

cos(A+B+C) = –1; cos(A+B) cosC – sin(A+B) sinC = –1

(cosA cosB – sinA sin B) cosC – sinC (sinA cosB + cosA sinB) = –1

1+ cosA cosB cosC= sinA sinB cosC + sinB sinC cosA + sinC sinA cosB

dividing by cosA cosB cosC

$$\frac{1+q}{q} = \sum \tan A \tan B$$

Hence (i) becomes $qx^3 - px^2 + (1 + q)x - p = 0$ Ans.

E. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC FUNCTIONS

(a) Min. value of $a^2 \tan^2\theta + b^2 \cot^2\theta = 2ab$

(b) Max and Min. value of $a\cos\theta + b\sin\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$

(c) If $f(\theta) = a\cos(\alpha+\theta) + b\cos(\beta+\theta)$ where a, b, α and β are known quantities then

$-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le f(\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$

(d) If $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$ and $\alpha+\beta = \sigma$ (constant) then the maximum values of the expression

$\cos\alpha\cos\beta$, $\cos\alpha + \cos\beta$, $\sin\alpha + \sin\beta$ and $\sin\alpha\sin\beta$

occurs when $\alpha=\beta=\sigma/2$

(e) If $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$ and $\alpha+\beta = \sigma$ (constant) then the minimum values of the expression

$\sec\alpha + \sec\beta$, $\tan\alpha + \tan\beta$, $\mathrm{cosec}\,\alpha + \mathrm{cosec}\,\beta$ occurs when $\alpha=\beta=\sigma/2$.

(f) If A, B, C are the anlges of a triangle then maximum value of

sinA + sinB + sinC and sinA sinB sinC occurs when A = B = C = 60º

(g) In case a quadratic in $\sin\theta$ or $\cos\theta$ is given then the maximum or minimum values can be interpreted by making a perfect square

Ex.48 Find the minimum vertical distance between the graphs of y = 2 + sin x and y = cos x.

Sol. $d_{min} = \min(2 + \sin x - \cos x) = \min[2 + \sqrt{2}\sin\left(x-\frac{\pi}{4}\right)] = 2 - \sqrt{2}$ at $x = \frac{7}{4}$

Ex.49 If $a\sin^2x + b$ lies in the interval $[-2, 8]$ for every $x \in R$ then find the value of (a – b).

Sol. $f(x) = a\sin^2x + b$

f (x) has a maximum value of 8 which occurs when $\sin^2x = 1$

$\therefore$ a + b = 8 ....(1)

|||ly f (x) has a minimum value of – 2 which occurs where sin x = 0

$\therefore$ b = – 2 ....(2)

from (1) and (2) a = 10; b = – 2 $\Rightarrow$ a – b = 12 [Ans. 12]

Ex.50 Find the greatest value of c such that system of equations

$x^2 + y^2 = 25$; x + y = c has a real solution.

Sol. put $x = 5\cos\theta$ $y = 5\sin\theta$

$\therefore$ $5(\cos\theta + \sin\theta) = c$; but $(\cos\theta + \sin\theta)_{max} = \sqrt{2}$ and $(\cos\theta + \sin\theta)_{min} = -\sqrt{2}$

hence, $c_{max} = 5\sqrt{2}$ Ans.

Ex.51 Find the minimum and maximum value of $f(x, y) = 7x^2 + 4xy + 3y^2$ subjected to $x^2 + y^2 = 1$.

Sol. Let $x = \cos\theta$ and $y = \sin\theta$

$y = f(\theta) = 7\cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta = 3 + 2\sin 2\theta + 2(1 + \cos 2\theta)$

$= 5 + 2(\sin 2\theta + \cos 2\theta)$ but $-\sqrt{2} \le (\sin 2\theta + \cos 2\theta) \le \sqrt{2}$

$\therefore$ $y_{max} = 5 + 2\sqrt{2}$ and $y_{min} = 5 - 2\sqrt{2}$

Ex.52 Let $f(x) = \sin^6x + \cos^6x + k(\sin^4x + \cos^4x)$ for some real number k. Determine
(a) all real numbers k for which $f(x)$ is constant for all values of x.
(b) all real numbers k for which there exists a real number 'c' such that f (c) = 0.

Sol. (a) $f(x) = (\sin^2x + \cos^2x)^3 - 3\sin^2\cos^2x(\sin^2x + \cos^2x) + k[(\sin^2 + \cos^2x)^2 - 2\sin^2x\cos^2x]$
$= 1 - 3\sin^2x\cos^2x + k(1 - 2\sin^2x\cos^2x)$
$f(x) = (k + 1) - \sin^2x\cos^2x\,(2k + 3)$ ....(1)

for f (x) to be independent of x $k = -\dfrac{3}{2}$ Ans.

(b) $f(c) = (k + 1) - \sin^2c\cos^2c\,(2k + 3) = 0$

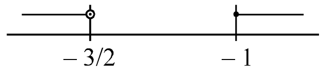
$\therefore \quad \sin^2c\cos^2c = \dfrac{k+1}{2k+3} \quad \Rightarrow \quad \dfrac{1}{4}(\sin^2 2c) = \dfrac{k+1}{2k+3} \quad \Rightarrow \quad \sin^2 2c = \dfrac{4(k+1)}{2k+3}$

but $0 \le \sin^2 2c \le 1$

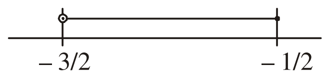
$\therefore \quad 0 \le \dfrac{4(k+1)}{2k+3} \le 1$

solving $\quad \dfrac{4(k+1)}{2k+3} \ge 0; \qquad \dfrac{(k+1)}{2k+3} \ge 0$

hence $\quad k \ge -1 \qquad$ or $\qquad k < -\dfrac{3}{2}$



again solving $\dfrac{4(k+1)}{2k+3} \le 1; \ \dfrac{4k+4}{2k+3} - 1 \le 0; \ \dfrac{4k+4-2k-3}{2k+3} \le 0$



$\dfrac{2k+1}{2k+3} \le 0 \qquad$ Hence $\qquad k \in \left[-1, -\dfrac{1}{2}\right]$ Ans.

Ex.53 If $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ are real numbers, show that,
$(\cos\alpha_1 + \cos\alpha_2 + \ldots\ldots + \cos\alpha_n)^2 + (\sin\alpha_1 + \ldots\ldots + \sin\alpha_n)^2 \le n^2$.

Sol. $LHS = (\cos^2\alpha_1 + \sin^2\alpha_1) + \ldots\ldots. + (\cos^2\alpha_n + \sin^2\alpha_n) + 2\,\underbrace{\Sigma\cos(\alpha_1 - \alpha_2)}_{{}^nC_2 \text{ terms}}$

$\le n + 2\dfrac{n(n-1)}{2} = n^2$

Ex.54 Show that the expression $\cos\theta\,(\sin\theta + \sqrt{\sin^2\theta + \sin^2\alpha})$ always lies between the values of $\pm\sqrt{1+\sin^2\alpha}$.

Sol. Let $y = \cos\theta\,(\sin\theta + \sqrt{\sin^2\theta + \sin^2\alpha})$

or, $\quad y - \cos\theta\sin\theta = \cos\theta\,(\sqrt{\sin^2\theta + \sin^2\alpha})$
or, $\quad (y - \cos\theta\sin\theta)^2 = \cos^2\theta\,(\sin^2\theta + \sin^2\alpha)$
or, $\quad y^2 - 2y\sin\theta\cos\theta + \cos^2\theta = \cos^2\theta\sin^2\theta + \cos^2\theta\sin^2\alpha$
or, $\quad y^2 - 2y\sin\theta\cos\theta + \cos^2\theta = \cos^2\theta + \cos^2\theta\,.\,\sin^2\alpha$
[Here we have added $\cos^2\theta$ on both sides to get $1 + \sin^2\alpha$]

or, $y^2 - 2y \sin\theta \cos\theta + \cos^2\theta = \cos^2\theta\,(1 + \sin^2\alpha)$

or, $y^2.\sec^2\theta - 2y\tan\theta + 1 = 1 + \sin^2\alpha$ (dividing by $\cos^2\theta$)

or, $y^2\tan^2\theta - 2y\tan\theta + 1 = (1 + \sin^2\alpha) - y^2$ ($\sec^2\theta = 1 + \tan^2\theta$)

or, $(y\tan\theta - 1)^2 = (1 + \sin^2\alpha) - y^2$

$\because$ square of a real number ≥ 0

$\therefore$ $1 + \sin^2\alpha - y^2 \geq 0$

or, $y^2 - (\sqrt{1+\sin^2\alpha})^2 \leq 0 \Rightarrow$ y lies between $-\sqrt{1+\sin^2\alpha}$ and $\sqrt{1+\sin^2\alpha}$.

F. SUMMATION OF TRIGONOMETRIC SERIES

Sum of sines or cosines of n angles

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha + 2\beta) + \ldots\ldots + \sin(\alpha + \overline{n-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \sin\left(\alpha + \frac{n-1}{2}\beta\right)$$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + \ldots\ldots + \cos(\alpha + \overline{n-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

Ex.55 Find the sum of the series, $\cos\frac{\pi}{2n+1} + \cos\frac{3\pi}{2n+1} + \cos\frac{5\pi}{2n+1} + \ldots\ldots..$ upto *n* terms.

Do not use any direct formula of summation.

Sol. Let $\theta = \frac{\pi}{2n+1}$

$S = \cos\theta + \cos 3\theta + \cos 5\theta + \ldots\ldots.. \cos(2n-1)\theta$

$(2\sin\theta)\,S = 2\sin\theta\,[\cos\theta + \cos 3\theta + \cos 5\theta + \ldots\ldots.. \cos(2n-1)\theta]$

$T_1 = \sin 2\theta - 0$; $T_2 = \sin 4\theta - \sin 2\theta$; $T_3 = \sin 6\theta - \sin 4\theta$;...........$T_n = \sin 2n\theta - \sin 2(n-1)\theta$

$(2\sin\theta)\,S = \sin 2n\theta$; $S = \dfrac{\sin\frac{2n\pi}{2n+1}}{2\sin\frac{\pi}{2n+1}} = \dfrac{1}{2}$ Ans.

Ex.56 Given $\sum_{k=1}^{35} \sin 5k = \tan\left(\frac{m}{n}\right)$, where angles are measured in degrees, and *m* and *n* are relatively prime positive integers that satisfy $\frac{m}{n} < 90$, find the value of (m + n).

Sol. LHS: $S = \sin 5 + \sin 10 + \sin 15 + \ldots\ldots\ldots. + \sin 170 + \sin 175$

$S\left(2\sin\frac{5}{2}\right) = 2\sin\frac{5}{2}\,[\sin 5 + \sin 10 + \ldots\ldots.. + \sin 175]$

$T_1 = \cos\frac{5}{2} - \cos\frac{15}{2}$; $T_2 = \cos\frac{15}{2} - \cos\frac{25}{2}$ ....; $T_{35} = \cos\frac{345}{2} - \cos\frac{355}{2}$

$\left(2\sin\frac{5}{2}\right)$ · S = $\cos\frac{5}{2} - \cos\frac{355}{2} = 2\sin\frac{180}{2}\cdot\sin\frac{175}{2} = 2\sin\frac{175}{2}$

$$S = \frac{\sin\frac{175}{2}}{\sin\frac{5}{2}} = \frac{\sin\frac{175}{2}}{\cos\left(90-\frac{5}{2}\right)} = \frac{\sin\frac{175}{2}}{\cos\frac{175}{2}} = \tan\left(\frac{175}{2}\right) = \tan\left(\frac{m}{n}\right)$$

$\therefore$ m = 175 and n = 2 $\Rightarrow$ m + n = 177 Ans.

Ex.57 Find the sum of the series,

$\cot 2x \cdot \cot 3x + \cot 3x \cdot \cot 4x + \ldots\ldots + \cot(n+1)x \cdot \cot(n+2)x$.

Sol. $\cot x = \cot[(n+2)x - (n+1)x] = \dfrac{\cot(n+2)x.\cot(n+1)x+1}{\cot(n+1)x-\cot(n+2)x}$

or $\cot x[\cot(n+1)x - \cot(n+2)x] = \cot(n+2)x \cdot \cot(n+1)x + 1$

Hence $\cot(n+1)x \cdot \cot(n+2)x = \cot x[\cot(n+1)x - \cot(n+2)x] - 1$

Put n = 1, 2, 3, ..., n and adding we get sum of the series

$$= \cot x[\cot 2x - \cot(n+2)x] - \frac{n}{2}$$

Ex.58 Evaluate : $\sum_{n=1}^{\infty}\left(\dfrac{\tan\frac{\theta}{2^n}}{2^{n-1}\,.\cos\frac{\theta}{2^{n-1}}}\right)$.

Sol. $T_1 = \dfrac{\tan\frac{\theta}{2}}{\cos\theta} = \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}\cos\theta} = \dfrac{2\sin^2\frac{\theta}{2}}{\sin\theta\cos\theta} = \dfrac{2(1-\cos\theta)}{\sin 2\theta} = \dfrac{2}{\sin 2\theta} - \dfrac{1}{\sin\theta}$

Similarly $T_2 = \dfrac{1}{\sin\theta} - \dfrac{1}{2\sin\frac{\theta}{2}}$; $T_n = \dfrac{1}{2^{n-2}\sin\frac{\theta}{2^{n-2}}} - \dfrac{1}{2^{n-1}\sin\frac{\theta}{2^{n-1}}}$ $\Rightarrow$ Ans. : $\dfrac{2}{\sin 2\theta} - \dfrac{1}{\theta}$

Ex.59 Let $f(x)$ denote the sum of the infinite trigonometric series, $f(x) = \sum_{n=1}^{\infty}\sin\frac{2x}{3^n}\sin\frac{x}{3^n}$.

Find $f(x)$ (independent of n) also evaluate the sum of the solutions of the equation f (x) = 0 lying in the interval (0, 629).

Sol. $f(x) = \sum_{n=1}^{\infty}\sin\frac{2x}{3^n}\sin\frac{x}{3^n} = \frac{1}{2}\sum_{n=1}^{\infty}2\sin\frac{2x}{3^n}\sin\frac{x}{3^n} = \frac{1}{2}\sum_{n=1}^{\infty}\left[\cos\frac{x}{3^n} - \cos\frac{x}{3^{n-1}}\right]$

now substituting n = 1, 2, 3, 4........

$$f(x) = \frac{1}{2}\left[\cos\frac{x}{3} - \cos x\right] + \frac{1}{2}\left[\cos\frac{x}{3^2} - \cos\frac{x}{3}\right] + \frac{1}{2}\left[\cos\frac{x}{3^3} - \cos\frac{x}{3^2}\right] \ldots\ldots\ldots + \frac{1}{2}\left[\cos\frac{x}{3^n} - \cos\frac{x}{3^{n-1}}\right]$$

$f(x) = \lim_{n\to\infty}\frac{1}{2}\left[\cos\frac{x}{3^n} - \cos x\right] = \frac{1}{2}[1 - \cos x]$ now f (x) = 0 $\Rightarrow$ cos x = 1 $\quad$ x = 2nπ, n ∈ I

sum of the solutions in (0, 629), S = 2[π + 2π + 3π + ... + 100π] = 2 · 5050π = 10100π Ans.

Ex.60 Evaluate $\sum_{n=1}^{89} \frac{1}{1+(\tan n°)^2}$.

Sol. $S = \frac{1}{1+(\tan 1°)^2} + \frac{1}{1+(\tan 2°)^2} + \frac{1}{1+(\tan 3°)^2} + ... + \frac{1}{1+(\tan 88°)^2} + \frac{1}{1+(\tan 89°)^2}$

reversing the sum

$S = \frac{1}{1+(\cot 1°)^2} + \frac{1}{1+(\cot 2°)^2} + ... + \frac{1}{1+(\cot 88°)^2} + \frac{1}{1+(\cot 89°)^2}$

$2S = \sum_{n=1}^{89} \frac{1}{1+(\tan n°)^2} + \frac{1}{1+(\cot n°)^2} = \sum_{n=1}^{89} \frac{1}{1+(\tan n°)^2} + \frac{(\tan n°)^2}{1+(\tan n°)^2}$

$= \sum_{n=1}^{89} 1 = 1 + 1 + ... + 1 = 89$ $\qquad \therefore \quad S = 44.5$ Ans.

G. ELIMINATION

Ex.61 Eliminate θ between the equation a $\sec\theta$ + b $\tan\theta$ + c = 0 and p $\sec\theta$ + q $\tan\theta$ + r = 0.

Sol. Given a $\sec\theta$ + b $\tan\theta$ + c = 0 ...(1)

and p $\sec\theta$ + q $\tan\theta$ + r = 0 ...(2)

Solving (1) and (2) by cross multiplication method, we have

$\frac{\sec\theta}{br - qc} = \frac{\tan\theta}{pc - ar} = \frac{1}{aq - pb}$ $\quad \because \quad \sec^2\theta - \tan^2\theta = 1$

$\therefore \quad \left(\frac{br - qc}{aq - pb}\right)^2 - \left(\frac{pc - ar}{aq - pb}\right)^2 = 1$

or, $(br - qc)^2 - (pc - ar)^2 = (aq - pb)^2$

Ex.62 If θ is eliminated from the equations, a $\cos\theta$ + b $\sin\theta$ = c & a $\cos^2\theta$ + b $\sin^2\theta$ = c, show that the eliminant is, $(a - b)^2 (a - c)(b - c) + 4a^2b^2 = 0$.

Sol. a $\cos\theta$ + b $\sin\theta$ = c ....(1)

a $\cos^2\theta$ + b $\sin^2\theta$ = c ....(2)

From (2) $\sin^2\theta = \frac{c - a}{b - a}$ and $\cos^2\theta = \frac{b - c}{b - a}$

Now squaring (1) $\quad a^2\cos^2\theta + b^2\sin^2\theta + 2ab\sin\theta\cos\theta = c^2$

$a^2\frac{b - c}{b - a} + b^2\frac{c - a}{b - a} - c^2 = -2ab\sqrt{\frac{b - c}{b - a}}\sqrt{\frac{c - a}{b - a}}$

or $a^2(b - c) + b^2(c - a) - c^2(b - a) = -2ab\sqrt{b - c}\sqrt{c - a}$

$(a - b)(b - c)(c - a) = 2ab\sqrt{b - c}\sqrt{c - a}$

$(a - b)^2(b - c)^2(c - a)^2 = 4a^2b^2(b - c)(c - a)$

$(a - b)^2(b - c)(c - a) = 4a^2b^2 \Rightarrow$ Result

Ex.63 Eliminate θ and φ from the relations

$m^2 \tan^2\theta + n^2 \tan^2 \varphi = 1$, $m^2 \cos^2 \theta + n^2 \sin^2 \varphi = 1$, $m \sin \theta = n \cos \varphi$.

and find the relationship between m and n.

Sol. In order to be able to take advantage of the third relation, rewrite the second so that it embodies the products m sin θ and n cos φ.

$m^2 \sin^2\theta + n^2 \cos^2 \varphi = m^2 + n^2 - 1$

Then, taking into account the third given equation, we get $2n^2 \cos^2\varphi = m^2 + n^2 - 1$

Further more, from the third relation we have

$$\cos^2 \theta = 1 - \sin^2 \theta = 1 - \frac{n^2}{m^2} \cos^2\varphi = 1 - \left(\frac{m^2 + n^2 - 1}{2m^2}\right) = \frac{m^2 - n^2 + 1}{2m^2}$$

Now rewrite the first of the given relation in the form

$$m^2\left(\frac{1}{\cos^2\theta} - 1\right) + n^2\left(\frac{1}{\cos^2\varphi} - 1\right) = 1$$

and substitute the expressions found for $\cos^2\theta$ and $\cos^2\varphi$ to obtain a relation between m and n:

$$\frac{2m^4}{m^2 - n^2 + 1} + \frac{2n^4}{m^2 + n^2 - 1} = m^2 + n^2 + 1$$

Miscellaneous Questions

Ex.64 Form a biquadratic equation whose roots are , $\cos\frac{\pi}{9}$, $\cos\frac{3\pi}{9}$, $\cos\frac{5\pi}{9}$, $\cos\frac{7\pi}{9}$.

Sol. Let $x_1 = \cos 20º$; $x_2 = \cos 60º$; $x_3 = \cos 100º$; $x_4 = \cos 140º$

Hence the equation is, $x^4 - (\Sigma x_1) x^3 + (\Sigma x_1 x_2) x^2 - (\Sigma x_1 x_2 x_3) x + x_1 x_2 x_3 x_4 = 0$

Now $\Sigma x_1 = \frac{1}{2} + \cos 20º - \cos 40º - \cos 80º$

$= \frac{1}{2} + \cos 20º - (2 \cos 60º \cos 20º) = \frac{1}{2}$

$x_1 x_2 x_3 x_4 = \frac{1}{2} \cos 20º \cos 40º \cos 80º = \frac{1}{16}$

$\Sigma x_1 x_2 = \cos 60º (\cos 20º + \cos 100º + \cos 140º) + \cos 20º \cos 100º + \cos 20º \cos 140º$
$+ \cos 100º \cos 140º$

$= \frac{1}{2}$ (zero) $+ \frac{1}{2}$ [cos 120º + cos 80º + cos 160º + cos 120º + cos 120º + cos 40º]

$$= \frac{1}{2}\left[-\frac{3}{2} + \cos 80° - \cos 20° + \cos 40°\right] = -\frac{3}{4}$$

$$\Sigma x_1 x_2 x_3 = \cos 20º \cos 60º \cos 100º \cos 140º \left[\frac{1}{\cos 60°} + \frac{1}{\cos 20°} + \frac{1}{\cos 100°} + \frac{1}{\cos 140°}\right]$$

$$= \frac{1}{16}\left[2 + \frac{1}{\cos 20°} - \left(\frac{1}{\cos 80°} + \frac{1}{\cos 40°}\right)\right] = \frac{1}{16}\left[2 + \frac{1}{\cos 20°} - \left(\frac{\cos 40° + \cos 80°}{\cos 40°\ \cos 80°}\right)\right]$$

$= \frac{1}{16}\left[2 + \frac{1}{\cos 20°} - \frac{\cos 20°}{\cos 40° \cos 80°}\right] \quad = \frac{1}{16}\left[2 + \frac{\cos 40° \cos 180° - \cos^2 20°}{\cos 20° . \cos 40° . \cos 80°}\right]$

$= \frac{1}{16}\left[2 + \frac{\frac{1}{2}(\cos 120° + \cos 40°) - \left(\frac{1+\cos 40°}{2}\right)}{1/8}\right] = \frac{1}{16}\left[2 + \frac{-\frac{1}{4} + \frac{1}{2}\cos 40° - \frac{1}{2} - \frac{1}{2}\cos 40°}{1/8}\right] = \frac{1}{16}\left[2 - \frac{3}{4} . \frac{8}{1}\right] = -\frac{1}{4}$

Hence the required equation is, $x^4 - \frac{1}{2}x^3 + \left(-\frac{3}{4}\right)x^2 - \left(-\frac{1}{4}\right)x + \frac{1}{16} = 0$

$\Rightarrow \quad 16x^4 - 8x^3 - 12x^2 + 4x + 1 = 0$

Ex.65 Prove that $\tan\frac{\pi}{7} . \tan\frac{2\pi}{7} . \tan\frac{3\pi}{7} = \sqrt{7}$

Sol. Let $\theta = \frac{\pi}{7}$ $\therefore$ $7\theta = \pi$

or, $4\theta + 3\theta = \pi$ or, $\tan(4\theta) = \tan(\pi - 3\theta)$ or, $\tan 4\theta = -\tan 3\theta$

or, $\frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = -\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$

or, $\frac{4z - 4z^3}{1 - 6z^2 + z^4} = -\left(\frac{3z - z^3}{1 - 3z^2}\right)$ [where $\tan\theta = z$ (suppose)]

or, $(4 - 4z^2)(1 - 3z^2) = -(3 - z^2)(1 - 6z^2 + z^4)$ or $12z^4 - 16z^2 + 4 = -(-z^6 + 9z^4 - 19z^2 + 3)$
or, $z^6 - 21z^4 + 35z^2 - 7 = 0$ ...(1)

This is cubic equation in z^2 i.e. in $\tan^2\theta$, the roots of this equation are therefore $\tan^2\frac{\pi}{7}$, $\tan^2\frac{2\pi}{7}$ and $\tan^2\frac{3\pi}{7}$

From (1), product of the roots = 7

$\Rightarrow \tan^2\frac{\pi}{7} . \tan^2\frac{2\pi}{7} . \tan^2\frac{3\pi}{7} = 7 \quad \Rightarrow \quad \tan\frac{\pi}{7} . \tan\frac{2\pi}{7} . \tan\frac{3\pi}{7} = \sqrt{7}$ Hence the result.

Ex.66 If $\theta = \frac{2\pi}{7}$, prove that $\tan\theta\tan 2\theta + \tan 2\theta \tan 4\theta + \tan 4\theta \tan\theta = -7$.

Sol. We have to prove that

$(\tan\theta\tan 2\theta + 1) + (\tan 2\theta\tan 4\theta + 1) + (\tan 4\theta \tan\theta + 1) = -4$

or, $\left(\frac{\sin\theta\sin 2\theta}{\cos\theta\cos 2\theta} + 1\right) + \left(\frac{\sin 2\theta\sin 4\theta}{\cos 2\theta\cos 4\theta} + 1\right) + \left(\frac{\sin 4\theta\sin\theta}{\cos 4\theta\cos\theta} + 1\right) = -4$

or, $\frac{\cos\theta}{\cos\theta\cos 2\theta} + \frac{\cos 2\theta}{\cos 2\theta\cos 4\theta} + \frac{\cos 3\theta}{\cos\theta\cos 4\theta} = -4$

or, $\frac{1}{\cos 2\theta} + \frac{1}{\cos 4\theta} + \frac{1}{\cos\theta} = -4 \quad \left[\therefore 3\theta = 2\pi - 4\theta \text{ as } \theta = \frac{2\pi}{7}\right]$

or, $\frac{\cos\theta\cos 4\theta + \cos 2\theta\cos\theta + \cos 4\theta\cos 2\theta}{\cos\theta\cos 2\theta\cos 4\theta} = -4$

or, $2\cos\theta\cos 4\theta + 2\cos 2\theta\cos\theta + 2\cos 4\theta\cos 2\theta = -8\cos\theta\cos 2\theta\cos 4\theta$

or, $\cos 5\theta + \cos 3\theta + \cos 3\theta + \cos\theta + \cos 6\theta + \cos 2\theta = -8\cos\theta \cos 2\theta \cos 4\theta$
or, $\cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + \cos 5\theta + \cos 6\theta = -8\cos\theta \cos 2\theta \cos 3\theta$
[$\therefore$ $3\theta = 2\pi - 4\theta$ $\quad\therefore\quad$ $\cos 3\theta = \cos 4\theta$]
or, $2\cos\theta + 2\cos 2\theta + 2\cos 3\theta = -8\cos\theta \cos 2\theta \cos 3\theta$
[$\because$ $6\theta = 2\pi - \theta$, $5\theta = 2\pi - 2\theta$ and $4\theta = 2\pi - 3\theta$]

or, $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -4\cos\frac{2\pi}{7} \cos\frac{4\pi}{7} \cos\frac{6\pi}{7}$

Now, L.H.S. $= \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$

Also $\cos\frac{2\pi}{7} \cdot \cos\frac{4\pi}{7} \cos\frac{6\pi}{7} = \frac{1}{8}$

$\therefore$ R.H.S. $= -4.\frac{1}{8} = -\frac{1}{2}$. Hence the result.

Ex.67 In a triangle ABC prove that $\frac{\sin A + \sin B + \sin C}{\cot A + \cot B + \cot C} \le \frac{3}{2}$.

Sol. Let $y = \frac{\sin A + \sin B + \sin C}{\cot A + \cot B + \cot C}$.

We know that in a triangle ABC, $\sum \cot A \cot B = 1$

$\Rightarrow$ $(\cot A + \cot B + \cot C)^2 = \sum \cot^2 A + 2 = \frac{1}{2}\sum(\cot A - \cot B)^2 + 3$

$\Rightarrow$ $(\cot A + \cot B + \cot C)^2 \ge 3 \Rightarrow$ $\cot A + \cot B + \cot C \ge \sqrt{3}$

Also $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$

$\Rightarrow$ y is maximum whose denominator is minimum and numerator is maximum simultaneously.

$\Rightarrow$ $y \le \frac{3\sqrt{3}}{2\sqrt{3}} \le \frac{3}{2}$.

Ex.68 In triangle ABC, $\cos A . \cos B + \cos B . \cos C + \cos C . \cos A = 1 - 2 \cos A . \cos B . \cos C$. Prove that it is possible if and only if ΔABC is equilateral.

Sol. $\Sigma \cos A . \cos B = 1 - 2\cos A . \cos B . \cos C = 1 - \cos C (\cos(A+B) + \cos(A-B))$
$= 1 - \cos C (\cos(A - B) - \cos C) = 1 + \cos(A + B)\cos(A - B) + \cos^2 C$
$= 1 + \cos^2 A - \sin^2 B + \cos^2 C = \cos^2 A + \cos^2 B + \cos^2 C = \Sigma \cos^2 A$.
Thus we have, $2\Sigma \cos^2 A - 2\Sigma \cos A . \cos B = 0$
$\Rightarrow (\cos A - \cos B)^2 + (\cos B - \cos C)^2 + (\cos C - \cos A)^2 = 0$ $\Rightarrow \cos A = \cos B = \cos C \Rightarrow \angle A = \angle B = \angle C$
Thus triangle ABC is equilateral

Now if Δ is equilateral $\angle A = \angle B = \angle C = \frac{\pi}{3} \Rightarrow \Sigma \cos A \cos B = \frac{3}{4}$ and $1 - 2\cos A \cos B \cos C$

$= 1 - \frac{2}{8} = \frac{3}{4}$. Hence the given expression is true if and only if Δ ABC is equilateral.

EXERCISE – I SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. If $\tan\alpha+\cot\alpha=a$ then the value of $\tan^4\alpha+\cot^4\alpha$ =
(A) $a^4 + 4a^2 + 2$ (B) $a^4 - 4a^2 + 2$
(C) $a^4 - 4a^2 - 2$ (D) None of these
Sol.

2. If $a\cos\theta + b\sin\theta = 3$ & $a\sin\theta - b\cos\theta = 4$ then $a^2 + b^2$ has the value =
(A) 25 (B) 14 (C) 7 (D) None of these
Sol.

3. The value of $\tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots\ldots \tan 89^\circ$ is
(A) 1 (B) 0 (C) ∞ (D) 1/2
Sol.

4. $$\frac{\tan\left(x-\frac{\pi}{2}\right)\cos\left(\frac{3\pi}{2}+x\right)-\sin^3\left(\frac{7\pi}{2}-x\right)}{\cos\left(x-\frac{\pi}{2}\right).\tan\left(\frac{3\pi}{2}+x\right)}$$

when simplified reduces to :
(A) $\sin x \cos x$ (B) $-\sin^2 x$ (C) $-\sin x \cos x$ (D) $\sin^2 x$
Sol.

5. The expression
$$3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi+\alpha)\right]$$
is equal to
(A) 0 (B) 1 (C) 3 (D) $\sin 4\alpha + \sin 6\alpha$
Sol.

6. $\cos(540^\circ - \theta) - \sin(630^\circ - \theta)$ is equal to
(A) 0 (B) $2\cos\theta$ (C) $2\sin\theta$ (D) $\sin\theta - \cos\theta$
Sol.

7. The value of $\sin(\pi+\theta)\sin(\pi-\theta)\,\text{cosec}^2\theta$ is equal to
(A) −1 (B) 0 (C) $\sin\theta$ (D) None of these
Sol.

8. If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$, then the value of $1 + \cot\alpha\tan\beta$ is
(A) 1 (B) −1 (C) 2 (D) None of these
Sol.

9. The value of $\dfrac{\sin 24^\circ\cos 6^\circ - \sin 6^\circ \sin 66^\circ}{\sin 21^\circ \cos 39^\circ - \cos 51^\circ \sin 69^\circ}$ is
(A) −1 (B) 1 (C) 2 (D) None of these
Sol.

10. If 3 sinα = 5 sinβ, then $\dfrac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

Sol.

11. If tan A and tan B are the roots of the quadratic equation $x^2 - ax + b = 0$, then the value of $\sin^2(A + B)$

(A) $\dfrac{a^2}{a^2+(1-b)^2}$ (B) $\dfrac{a^2}{a^2+b^2}$

(C) $\dfrac{a^2}{(b+c)^2}$ (D) $\dfrac{a^2}{b^2(1-a)^2}$

Sol.

12. In a triangle ABC if tan A < 0 then :

(A) tan B . tan C > 1 (B) tan B . tan C < 1

(C) tan B . tan C = 1 (D) None of these

Sol.

13. If tan A – tan B = x and cot B – cot A = y, then cot (A – B) is equal to

(A) $\dfrac{1}{y}-\dfrac{1}{x}$ (B) $\dfrac{1}{x}-\dfrac{1}{y}$ (C) $\dfrac{1}{x}+\dfrac{1}{y}$ (D) None of these

Sol.

14. If tan 25º=x, then $\dfrac{\tan 155^\circ - \tan 115^\circ}{1+\tan 155^\circ \tan 115^\circ}$ is equal to

(A) $\dfrac{1-x^2}{2x}$ (B) $\dfrac{1+x^2}{2x}$ (C) $\dfrac{1+x^2}{1-x^2}$ (D) $\dfrac{1-x^2}{1+x^2}$

Sol.

15. If A + B = 225º, then the value of $\left(\dfrac{\cot A}{1+\cot A}\right).\left(\dfrac{\cot B}{1+\cot B}\right)$ is

(A) 2 (B) 1/2 (C) 3 (D) 1/3

Sol.

16. The value of tan 3A – tan 2A – tan A is equal to

(A) tan 3A tan 2A tan A

(B) – tan 3A tan 2A tan A

(C) tan A tan 2A – tan 2A tan 3A – tan 3A tan A

(D) None of these

Sol.

17. tan 203º + tan 22º + tan 203º tan 22º =
(A) –1 (B) 0 (C) 1 (D) 2
Sol.

18. The value of $\frac{1-\tan^2 15^\circ}{1+\tan^2 15^\circ}$ is
(A) 1 (B) $\sqrt{3}$ (C) $\frac{\sqrt{3}}{2}$ (D) 2
Sol.

19. If A lies in the third quadrant and 3 tan A – 4 = 0, then 5 sin 2A + 3 sinA + 4 cosA is equal to
(A) 0 (B) $-\frac{24}{5}$ (C) $\frac{24}{5}$ (D) $\frac{48}{5}$
Sol.

20. $\frac{\cos 20^\circ + 8\sin 70^\circ \sin 50^\circ \sin 10^\circ}{\sin^2 80^\circ}$ is equal to
(A) 1 (B) 2 (C) 3/4 (D) None of these
Sol.

21. If cos A = 3/4, then the value of
16cos^2 (A/2) – 32 sin (A/2) sin (5A/2) is
(A) –4 (B) –3 (C) 3 (D) 4
Sol.

22. The value of the expression
$\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)$ is
(A) 1/8 (B) 1/16 (C) 1/4 (D) 0
Sol.

23. The numerical value of sin 12º . sin 48º . sin 54º is equal to
(A) 1/2 (B) 1/4 (C) 1/16 (D) 1/8

Sol.

24. If A = tan 6º tan 42º and B = cot 66º cot 78º, then
(A) A = 2B (B) A = 1/3 B (C) A = B (D) 3A = 2B
Sol.

25. If $\alpha+\beta+\gamma = 2\pi$, then

(A) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(B) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$

(C) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = -\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(D) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 0$

Sol.

26. $\cos 0 + \cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{6\pi}{7} =$
(A) 1/2 (B) –1/2 (C) 0 (D) 1
Sol.

27. A regular hexagon & a regular dodecagon are inscribed in the same circle. If the side of the dodecagon is ($\sqrt{3}-1$), then the side of the hexagon is
(A) $\sqrt{2}+1$ (B) $\frac{\sqrt{3}+1}{2}$ (C) 2 (D) $\sqrt{2}$
Sol.

28. In a right angled triangle the hypotenuse is $2\sqrt{2}$ times the perpendicular drawn from the opposite vertex. Then the other acute angles of the triangle are

(A) $\frac{\pi}{3}$ & $\frac{\pi}{6}$ (B) $\frac{\pi}{8}$ & $\frac{3\pi}{8}$ (C) $\frac{\pi}{4}$ & $\frac{\pi}{4}$ (D) $\frac{\pi}{5}$ & $\frac{3\pi}{10}$

Sol.

29. If $\alpha \in \left[\frac{\pi}{2}, \pi\right]$ then the value of

$\sqrt{1+\sin\alpha} - \sqrt{1-\sin\alpha}$ is equal to

(A) $2\cos\frac{\alpha}{2}$ (B) $2\sin\frac{\alpha}{2}$ (C) 2 (D) None of these

Sol.

30. $\frac{1}{\cos 290^{\circ}} + \frac{1}{\sqrt{3}\sin 250^{\circ}} =$

(A) $\frac{2\sqrt{3}}{3}$ (B) $\frac{4\sqrt{3}}{3}$ (C) $\sqrt{3}$ (D) None of these

Sol.

31. If A + B + C = π & $\sin\left(A+\frac{C}{2}\right) = k\sin\frac{C}{2}$, then $\tan\frac{A}{2}\tan\frac{B}{2} =$

(A) $\frac{k-1}{k+1}$ (B) $\frac{k+1}{k-1}$ (C) $\frac{k}{k+1}$ (D) $\frac{k-1}{k}$

Sol.

32. The value of cot x + cot(60º + x) + cot (120º + x) is equal to

(A) cos 3x (B) tan 3x

(C) 3 tan 3x (D) $\frac{3-9\tan^2 x}{3\tan x - \tan^3 x}$

Sol.

33. If $x \in \left(\pi, \frac{3\pi}{2}\right)$ then

$4\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right) + \sqrt{4\sin^4 x + \sin^2 2x}$ is always equal to

(A) 1 (B) 2 (C) −2 (D) None of these

Sol.

34. In any triangle ABC, which is not right angled Σ cos A . cosec B . cosec C is equal to

(A) 1 (B) 2 (C) 3 (D) None of these

Sol.

35. If 3 cos x + 2 cos 3x = cos y, 3 sin x + 2 sin 3x = sin y, then the value of cos 2x is

(A) −1 (B) 1/8 (C) −1/8 (D) 7/8

Sol.

36. The expression $\frac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x}$ is equal to

(A) cos 2x (B) 2 cos x (C) $\cos^2 x$ (D) 1 + cos x

Sol.

37. If cos (A – B) = 3/5 and tan A tan B = 2,

(A) cosA cosB = $-\frac{1}{5}$ (B) sinA sinB = $-\frac{2}{5}$

(C) cos (A + B) = $-\frac{1}{5}$ (D) sin A cos B = $\frac{4}{5}$

Sol.

38. If $\cos\alpha + \cos\beta = a$, $\sin\alpha + \sin\beta = b$ and $\alpha - \beta = 2\theta$, then $\frac{\cos 3\theta}{\cos\theta}$ =

(A) $a^2 + b^2 - 2$ (B) $a^2 + b^2 - 3$

(C) $3 - a^2 - b^2$ (D) $(a^2 + b^2)/4$

Sol.

39. If A + B + C = $\frac{3\pi}{2}$, then cos 2A + cos 2B + cos 2C is equal to

(A) 1–4cos A cosB cosC (B) 4 sinA sinB sinC

(C) 1+2 cosA cosB cosC (D) 1–4 sinA sinB sinC

Sol.

40. If A + B + C = π & cos A = cos B . cos C then tan B . tanC has the value equal to

(A) 1 (B) 1/2 (C) 2 (D) 3

Sol.

41. For $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\frac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta}$ lies in the interval

(A) $(-\infty, \infty)$ (B) (–2, 2) (C) $(0, \infty)$ (D) (–1, 1)

Sol.

42. If $0 < x < \pi$ and cos x + sin x = $\frac{1}{2}$, then tan x is

(A) $\frac{(4-\sqrt{7})}{3}$ (B) $-\frac{(4+\sqrt{7})}{3}$

(C) $\frac{(1+\sqrt{7})}{4}$ (D) $\frac{(1-\sqrt{7})}{4}$

Sol.

43. Let α, β be such that $\pi < \alpha - \beta < 3\pi$.

If $\sin\alpha + \sin\beta = -\frac{21}{65}$ and $\cos\alpha + \cos\beta = -\frac{27}{65}$, then the value of $\cos\frac{\alpha-\beta}{2}$ is

(A) $-\frac{3}{\sqrt{130}}$ (B) $\frac{3}{\sqrt{130}}$ (C) $\frac{6}{65}$ (D) $-\frac{6}{65}$

Sol.

44. The value of the expression
cos 1° cos 2° .... cos 179° equals

(A) 0 (B) 1 (C) $1/\sqrt{2}$ (D) – 1

Sol.

45. Which is correct one ?

(A) sin 1° < sin 1 (B) sin 1° = sin 1

(C) sin 1° > sin 1 (D) sin 1° = sin $\frac{\pi}{180}$

Sol.

46. The value of cos 10° – sin 10° is

(A) Positive (B) Negative (C) 0 (D) 1

Sol.

47. The value of $\tan\frac{\pi}{16} + 2\tan\frac{\pi}{8} + 4$ is equal to

(A) $\cot\frac{\pi}{8}$ (B) $\cot\frac{\pi}{16}$ (C) $\cot\frac{\pi}{16}-4$ (D) None of these

Sol.

48. The value of $\cos\frac{\pi}{19}+\cos\frac{3\pi}{19}+\cos\frac{5\pi}{19}+\ldots+\cos\frac{17\pi}{19}$ is equal to

(A) 1/2 (B) 0 (C) 1 (D) None of these

Sol.

49. If $\frac{3\pi}{4} < \alpha < \pi$, then $\sqrt{2\cot\alpha+\frac{1}{\sin^2\alpha}}$ is equal to

(A) 1 + cot α (B) −1 − cot α

(C) 1 − cot α (D) −1 + cot α

Sol.

50. If $f(\theta) = \sin^4 \theta + \cos^2 \theta$, then range of $f(\theta)$ is

(A) $\left[\frac{1}{2}, 1\right]$ (B) $\left[\frac{1}{2}, \frac{3}{4}\right]$ (C) $\left[\frac{3}{4}, 1\right]$ (D) None of these

Sol.

51. If 2 cos x + sin x = 1, then value of 7 cos x + 6 sin x is equal to

(A) 2 or 6 (B) 1 or 3 (C) 2 or 3 (D) None of these

Sol.

52. If cosec A + cot A = $\frac{11}{2}$, then tan A is

(A) $\frac{21}{22}$ (B) $\frac{15}{16}$ (C) $\frac{44}{117}$ (D) $\frac{117}{43}$

Sol.

53. If 0° < x < 90° & cos x = $\frac{3}{\sqrt{10}}$, then the value of $\log_{10} \sin x + \log_{10} \cos x + \log_{10} \tan x$ is

(A) 0 (B) 1 (C) –1 (D) None of these

Sol.

54. If $\cot \alpha + \tan \alpha = m$ and $\frac{1}{\cos \alpha} - \cos \alpha = n$, then

(A) $m(mn^2)^{1/3} - n(nm^2)^{1/3} = 1$

(B) $m(m^2n)^{1/3} - n(nm^2)^{1/3} = 1$

(C) $n(mn^2)^{1/3} - m(nm^2)^{1/3} = 1$

(D) $n(m^2n)^{1/3} - m(mn^2)^{1/3} = 1$

Sol.

55. If $2\sec^2 \alpha - \sec^4 \alpha - 2\operatorname{cosec}^2 \alpha + \operatorname{cosec}^4 \alpha = 15/4$, then $\tan \alpha$ is equal to

(A) $1/\sqrt{2}$ (B) 1/2 (C) $1/2\sqrt{2}$ (D) 1/4

Sol.

56. If $\frac{\sin A}{\sin B} = \frac{\sqrt{3}}{2}$ and $\frac{\cos A}{\cos B} = \frac{\sqrt{5}}{2}$, 0 < A, B < $\pi/2$, then tan A + tan B is equal to

(A) $\sqrt{3}/\sqrt{5}$ (B) $\sqrt{5}/\sqrt{3}$ (C) 1 (D) $(\sqrt{5}+\sqrt{3})/\sqrt{5}$

Sol.

57. If 3 sin x + 4 cos x = 5 then 4 sin x – 3 cos x is equal to

(A) 0 (B) 1 (C) 5 (D) None of these

Sol.

58. If sin 2θ = k, then the value of $\frac{\tan^3\theta}{1+\tan^2\theta} + \frac{\cot^3\theta}{1+\cot^2\theta}$ is equal to

(A) $\frac{1-k^2}{k}$ (B) $\frac{2-k^2}{k}$ (C) $k^2 + 1$ (D) $2 - k^2$

Sol.

59. If $f(\theta) = \sin^2\theta + \sin^2\left(\theta + \frac{2\pi}{3}\right) + \sin^2\left(\theta + \frac{4\pi}{3}\right)$, then $f\left(\frac{\pi}{15}\right)$ is equal to

(A) $\frac{2}{3}$ (B) $\frac{3}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$

Sol.

EXERCISE – II MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. The value of $\frac{\sin x + \cos x}{\cos^3 x}$ =

(A) $1+\tan x + \tan^2 x - \tan^3 x$ (B) $1+\tan x+\tan^2 x+\tan^3 x$

(C) $1-\tan x + \tan^2 x + \tan^3 x$ (D) $(1 + \tan x) \sec^2 x$

Sol.

2. If (sec A + tan A) (sec B + tan B) (sec C + tan C) = (sec A – tan A) (sec B – tan B) (sec C – tan C) then each side is equal to

(A) 1 (B) –1 (C) 0 (D) None of these

Sol.

3. The value of $\frac{(\cos 11^\circ + \sin 11^\circ)}{(\cos 11^\circ - \sin 11^\circ)}$ is

(A) $-\tan 304^\circ$ (B) $\tan 56^\circ$ (C) $\cot 214^\circ$ (D) $\cot 34^\circ$

Sol.

4. If $\tan^2 \theta = 2 \tan^2 \phi + 1$, then the value of $\cos 2\theta + \sin^2 \phi$ is

(A) 1 (B) 2 (C) –1 (D) Independent of ϕ

Sol.

5. The value of $\cos\frac{\pi}{10}\cos\frac{2\pi}{10}\cos\frac{4\pi}{10}\cos\frac{8\pi}{10}\cos\frac{16\pi}{10}$ is

(A) $\frac{\sqrt{10+2\sqrt{5}}}{64}$ (B) $-\frac{\cos(\pi/10)}{16}$

(C) $\frac{\cos(\pi/10)}{16}$ (D) $-\frac{\sqrt{10+2\sqrt{5}}}{64}$

Sol.

6. If x + y = z, then $\cos^2 x + \cos^2 y + \cos^2 z - 2 \cos x \cos y \cos z$ is equal to

(A) $\cos^2 z$ (B) $\sin^2 z$ (C) $\cos (x + y - z)$ (D) 1

Sol.

7. If tan A + tan B + tan C = tan A . tan B . tan C, then
(A) A, B, C may be angles of a triangle
(B) A + B + C is an integral multiple of π
(C) sum of any two of A, B, C is equal to third
(D) None of these
Sol.

8. In a triangle tan A + tan B + tan C = 6 and tan A tan B = 2, then the values of tanA, tan B and tan C are
(A) 1, 2, 3 (B) 2, 1, 3
(C) 1, 2, 0 (D) None of these
Sol.

9. An extreme value of $1 + 4 \sin\theta + 3 \cos\theta$ is
(A) – 3 (B) – 4 (C) 5 (D) 6

Sol.

10. If the sides of a right angled triangle are $\{\cos 2\alpha + \cos 2\beta + 2\cos(\alpha+\beta)\}$ and $\{\sin 2\alpha + \sin 2\beta + 2\sin(\alpha+\beta)\}$, then the length of the hypotneuse is
(A) $2\,[1 + \cos(\alpha-\beta)]$ (B) $2\,[1 - \cos(\alpha-\beta)]$
(C) $4\cos^2\dfrac{\alpha-\beta}{2}$ (D) $4\sin^2\dfrac{\alpha+\beta}{2}$
Sol.

11. For $0 < \theta < \pi/2$, $\tan\theta + \tan 2\theta + \tan 3\theta = 0$ if
(A) $\tan\theta = 0$ (B) $\tan 2\theta = 0$
(C) $\tan 3\theta = 0$ (D) $\tan\theta \tan 2\theta = 2$
Sol.

12. $(a+2)\sin\alpha + (2a-1)\cos\alpha = (2a+1)$ if $\tan\alpha =$
(A) 3/4 (B) 4/3 (C) $\dfrac{2a}{a^2+1}$ (D) $\dfrac{2a}{a^2-1}$

Sol.

13. If $\tan x = \dfrac{2b}{a-c}$, $(a \neq c)$

$y = a\cos^2 x + 2b \sin x \cos x + c\sin^2 x$

$z = a\sin^2 x - 2b \sin x \cos x + c\cos^2 x$, then

(A) $y = z$ (B) $y + z = a + c$

(C) $y - z = a - c$ (D) $y - z = (a - c)^2 + 4b^2$

Sol.

14. $\left(\dfrac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\dfrac{\sin A + \sin B}{\cos A - \cos B}\right)^n$

(A) $2\tan^n \dfrac{A-B}{2}$ (B) $2\cot^n \dfrac{A-B}{2}$: n is even

(C) 0 : n is odd (D) None of these

Sol.

15. The equation $\sin^6 x + \cos^6 x = a^2$ has real solution if

(A) $a \in (-1,1)$ (B) $a \in (-1, -1/2)$

(C) $a \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ (D) $a \in (1/2, 1)$

Sol.

16. If $3 \sin\beta = \sin(2\alpha+\beta)$, then $\tan(\alpha+\beta) - 2\tan\alpha$ is

(A) independent of α (B) independent of β

(C) dependent of both α and β

(D) independent of α but dependent of β

Sol.

EXERCISE – III SUBJECTIVE QUESTIONS

1. Prove that

(i) $(\text{cosec}\,\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$

Sol.

(ii) $\dfrac{2\sin\theta\tan\theta(1-\tan\theta)+2\sin\theta\sec^2\theta}{(1+\tan\theta)^2} = \dfrac{2\sin\theta}{(1+\tan\theta)}$

Sol.

(iii) $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = |\sec A - \tan A|$

Sol.

2. Prove that

(i) $\dfrac{\cos A\,\text{cosec}A - \sin A\sec A}{\cos A + \sin A} = \text{cosec}\,A - \sec A$

Sol.

(ii) $\dfrac{1}{\sec\alpha - \tan\alpha} - \dfrac{1}{\cos\alpha} = \dfrac{1}{\cos\alpha} - \dfrac{1}{\sec\alpha + \tan\alpha}$

Sol.

(iii) $\dfrac{\cos^3 A + \sin^3 A}{\cos A + \sin A} + \dfrac{\cos^3 A - \sin^3 A}{\cos A - \sin A} = 2$

Sol.

3. Eliminate θ from the relations $a\sec\theta = 1 - b\tan\theta$, $a^2\sec^2\theta = 5 + b^2\tan^2\theta$

Sol.

4. Prove that :

(i) $\sin^2\dfrac{\pi}{6} + \cos^2\dfrac{\pi}{3} - \tan^2\dfrac{\pi}{4} = -\dfrac{1}{2}$

Sol.

(ii) $2\sin^2\frac{\pi}{6} + \text{cosec}\frac{7\pi}{6}\cos^2\frac{\pi}{3} = 0$

Sol.

(iii) $3\cos^2\frac{\pi}{4} + \sec\frac{2\pi}{3} + 5\tan^2\frac{\pi}{3} = \frac{29}{2}$

Sol.

5. Prove that :

(i) $\cot^2\frac{\pi}{6} + \text{cosec}\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6} = 6$

Sol.

(ii) $2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3} = 10$

Sol.

(iii) $\cos^2\frac{\pi}{5} + \sin^2\frac{\pi}{10} = \frac{3}{4}$

Sol.

6. Prove that : $\dfrac{\cos(\pi+\theta)\cos(-\theta)}{\sin(\pi-\theta)\cos\left(\frac{\pi}{2}+\theta\right)} = \cot^2\theta$.

Sol.

7. If $\tan\theta = -5/12$, θ is not in the second quadrant, then show that $\dfrac{\sin(360^\circ-\theta)+\tan(90^\circ+\theta)}{-\sec(270^\circ+\theta)+\text{cosec}(-\theta)} = \dfrac{181}{338}$

Sol.

8. Show that :

(i) $\sin 20^\circ \cdot \cos 40^\circ + \cos 20^\circ \cdot \sin 40^\circ = \sqrt{3}/2$

Sol.

(ii) $\cos 100^\circ \cdot \cos 40^\circ + \sin 100^\circ \cdot \sin 40^\circ = 1/2$

Sol.

9. Show that :

(i) $\sin^2 75^\circ - \sin^2 15^\circ = \sqrt{3}/2$

Sol.

(ii) $\sin^2 45^o - \sin^2 15^o = \sqrt{3}/4$

Sol.

10. Show that : $\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right) = \left(\frac{1}{\sqrt{2}}\right)\sin A$

Sol.

11. Show that :

$\cos 2\theta \cos \frac{\theta}{2} - \cos 3\theta \cos \frac{9\theta}{2} = \sin 5\theta \sin \frac{5\theta}{2}$.

Sol.

12. Prove that

$$\left\{\frac{1-\cot^2\left(\frac{\alpha-\pi}{4}\right)}{1+\cot^2\left(\frac{\alpha-\pi}{4}\right)} + \cos\frac{\alpha}{2}\cot 4\alpha\right\} \sec \frac{9\alpha}{2} = \operatorname{cosec} 4\alpha.$$

Sol.

13. Prove that $\frac{1}{\tan 3\alpha - \tan\alpha} - \frac{1}{\cot 3\alpha - \cot\alpha} = \cot 2\alpha$.

Sol.

14. Prove that

(i) $\frac{\sin^2 A - \sin^2 B}{\sin A\cos A - \sin B\cos B} = \tan(A+B)$

Sol.

(ii) $\cot(A + 15^o) - \tan(A - 15^o) = \frac{4\cos 2A}{1+2\sin 2A}$

Sol.

15. Prove than

(i) $\dfrac{\sec 8A - 1}{\sec 4A - 1} = \dfrac{\tan 8A}{\tan 2A}$

Sol.

(ii) $\dfrac{\cos A + \sin A}{\cos A - \sin A} - \dfrac{\cos A - \sin A}{\cos A + \sin A} = 2 \tan 2A$

Sol.

16. If A+B=45º, prove that (1+tan A) (1+tan B) = 2 and hence deduce that tan $22\dfrac{1^{\circ}}{2} = \sqrt{2} - 1$

Sol.

17. If $0 < \theta < \pi/4$, then show that

$\sqrt{2 + \sqrt{2(1 + \cos 4\theta)}} = 2 \cos \theta$.

Sol.

18. Prove that $\dfrac{\cos^3 A - \cos 3A}{\cos A} + \dfrac{\sin^3 A + \sin 3A}{\sin A} = 3$

Sol.

19. Find the value of

(i) 4 sin 18º cos 36º

Sol.

(ii) $\cos^2 72^{\circ} - \sin^2 54^{\circ}$

Sol.

20. Prove that

$\tan \theta \tan (60^{\circ} + \theta) \tan (60^{\circ} - \theta) = \tan 3\theta$ and hence deduce that tan 20º tan 40º tan 60º tan 80º = 3.

Sol.

21. Prove that $4(\cos^3 20^\circ+\cos^3 40^\circ)=3(\cos 20^\circ+\cos 40^\circ)$

Sol.

22. Prove that

(i) $\frac{\tan 3x}{\tan x} = \frac{2\cos 2x + 1}{2\cos 2x - 1}$

Sol.

(ii) $\frac{2\sin x}{\sin 3x} + \frac{\tan x}{\tan 3x} = 1$

Sol.

23. Prove that

(i) $\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7} = \frac{1}{8}$

Sol.

(ii) $\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11} = \frac{1}{32}$

Sol.

24. Prove that

$\sin^2\theta+\sin^2 2\theta+\sin^2 3\theta+....+\sin^2 n\theta=\frac{n}{2}-\frac{\sin n\theta\cos(n+1)\theta}{2\sin\theta}$

Sol.

25. If ϕ is the exterior angle of a regular polygon of n sides and θ is any constant, then prove that
$\sin\theta + \sin(\theta+2\phi) + ...$ up to n terms = 0

Sol.

26. If $x + y + z = \frac{\pi}{2}$ show that,

$\sin 2x + \sin 2y + \sin 2z = 4 \cos x \cos y \cos z$.

Sol.

27. If $x + y = \pi + z$, then prove that
$\sin^2 x + \sin^2 y - \sin^2 z = 2 \sin x \sin y \cos z$.

Sol.

28. If $A + B + C = 2S$ then prove that
$\cos(S - A) + \cos(S - B) + \cos(S - C) + \cos S = 4 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

Sol.

29. If $A + B + C = 0º$ then prove that
$\sin 2A + \sin 2B + \sin 2C = -4 \sin A \sin B \sin C$.

Sol.

30. Find the extreme values of

$\cos x \cos\left(\frac{2\pi}{3}+x\right) \cos\left(\frac{2\pi}{3}-x\right)$

Sol.

31. Find the maximum and minimum values of

(i) $\cos 2x + \cos^2 x$

Sol.

(ii) $\cos^2\left(\frac{\pi}{4}+x\right)(\sin x - \cos x)^2$

Sol.

32. Prove that, $\sin 3x \cdot \sin^3 x + \cos 3x \cdot \cos^3 x = \cos^3 2x$.

Sol.

33. If $\tan\beta = \frac{\tan\alpha+\tan\gamma}{1+\tan\alpha.\tan\gamma}$, prove than

$\sin 2\beta = \frac{\sin 2\alpha+\sin 2\gamma}{1+\sin 2\alpha.\sin 2\gamma}$.

Sol.

34. Show that :

(i) $\cot 7\frac{1^\circ}{2}$ or $\tan 82\frac{1^\circ}{2} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$

or $\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

Sol.

(ii) $\tan 142\frac{1}{2}^{\circ} = 2 + \sqrt{2} - \sqrt{3} - \sqrt{6}$.

Sol.

35. If sin x + sin y = a & cos x + cos y = b, show that,

$\sin(x + y) = \frac{2ab}{a^2 + b^2}$ and $\tan\frac{x - y}{2} = \pm\sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}$.

Sol.

36. Calculate the following without using trigonometric tables :

(i) tan 9º – tan 27º – tan 63º + tan 81º

Sol.

(ii) cosec 10º – $\sqrt{3}$ sec 10º

Sol.

(iii) $2\sqrt{2}\ \sin 10º \left[\dfrac{\sec 5º}{2} + \dfrac{\cos 40º}{\sin 5º} - 2\sin 35º\right]$

Sol.

(iv) cot 70º + 4 cos 70º

Sol.

(v) tan 10º – tan 50º + tan 70º

Sol.

37. If $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = \dfrac{-3}{2}$, prove that

$\cos\alpha + \cos\beta + \cos\gamma = 0$, $\sin\alpha + \sin\beta + \sin\gamma = 0$

Sol.

38. If $\dfrac{ax}{\cos\theta} + \dfrac{by}{\sin\theta} = a^2 - b^2$, $\dfrac{ax\sin\theta}{\cos^2\theta} - \dfrac{by\cos\theta}{\sin^2\theta} = 0$.

Show that $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

Sol.

39. If $P_n = \cos^n\theta + \sin^n\theta$ and $Q_n = \cos^n\theta - \sin^n\theta$, then show that $P_n - P_{n-2} = -\sin^2\theta \cos^2\theta\, P_{n-4}$
$Q_n - Q_{n-2} = -\sin^2\theta \cos^2\theta\, Q_{n-4}$ and hence show that
$P_4 = 1 - 2\sin^2\theta \cos^2\theta$
$Q_4 = \cos^2\theta - \sin^2\theta$
Sol.

40. If $\sin(\theta+\alpha) = a$ & $\sin(\theta+\beta) = b$ $(0 < \alpha, \beta, \theta < \pi/2)$ then find the value of $\cos 2(\alpha-\beta) - 4ab\cos(\alpha-\beta)$
Sol.

41. If $A + B + C = \pi$, Prove that
$\tan B \tan C + \tan C \tan A + \tan A \tan B = 1 + \sec A . \sec B . \sec C$.
Sol.

42. If $\tan^2 \alpha + 2 \tan\alpha \, . \, \tan 2\beta = \tan^2\beta + 2 \tan\beta \, . \, \tan 2\alpha$, then prove that each side is equal to 1 or $\tan \alpha = \pm \tan \beta$.

Sol.

43. For all θ in $\left[0, \frac{\pi}{2}\right]$ show that $\cos(\sin\theta) > \sin(\cos\theta)$

Sol.

44. Find the length of an arc of a circle of radius 10 cm which subtends an angle of 45° at the centre.

Sol.

45. If the arcs of the same length in two circles subtend angles 75° and 120° at the centre, find the ratio of their radii.

Sol.

46. If $\tan x = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2}$, find the value of $\sin \frac{x}{2}$ and $\cos \frac{x}{2}$.

Sol.

47. Prove that :

(i) $\sec^4 A (1 - \sin^4 A) - 2 \tan^2 A = 1$

Sol.

(ii) $\frac{\cot^2\theta(\sec\theta - 1)}{1 + \sin\theta} = \sec^2\theta . \frac{1 - \sin\theta}{1 + \sec\theta}$

Sol.

48. In a ΔABC, prove that

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

Sol.

EXERCISE – IV ADVANCED SUBJECTIVE QUESTIONS

1. Prove that : $\cos^2\alpha + \cos^2(\alpha+\beta) - 2\cos\alpha\ \cos\beta\ \cos(\alpha+\beta) = \sin^2\beta$

Sol.

2. Prove that : $\cos 2\alpha = 2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\ \sin\beta + \cos 2(\alpha+\beta)$

Sol.

3. Prove that :

(a) $\tan 20° \cdot \tan 40° \cdot \tan 60° \cdot \tan 80° = 3$

Sol.

(b) $\sin^4\frac{\pi}{16}+\sin^4\frac{3\pi}{16}+\sin^4\frac{5\pi}{16}+\sin^4\frac{7\pi}{16}=\frac{3}{2}$

Sol.

4. Calculate without using trigonometric tables :

(a) $4\cos 20° - \sqrt{3}\cot 20°$

Sol.

(b) $\dfrac{2\cos 40°-\cos 20°}{\sin 20°}$

Sol.

(c) $\cos^6 \dfrac{\pi}{16} + \cos^6 \dfrac{3\pi}{16} + \cos^6 \dfrac{5\pi}{16} + \cos^6 \dfrac{7\pi}{16}$

Sol.

(d) tan 10º – tan 50º + tan 70º

Sol.

5. If $X = \sin\left(\theta+\dfrac{7\pi}{12}\right) + \sin\left(\theta-\dfrac{\pi}{12}\right) + \sin\left(\theta+\dfrac{3\pi}{12}\right)$,

$Y = \cos\left(\theta+\dfrac{7\pi}{12}\right) + \cos\left(\theta-\dfrac{\pi}{12}\right) + \cos\left(\theta+\dfrac{3\pi}{12}\right)$

then prove that $\dfrac{X}{Y}-\dfrac{Y}{X} = 2\tan 2\theta$.

Sol.

6. Find the positive integers p, q, r, s satisfying $\tan\dfrac{\pi}{24} = (\sqrt{p}-\sqrt{q})(\sqrt{r}-s)$.

Sol.

7. If $m\tan(\theta - 30°) = n\tan(\theta + 120°)$, show that $\cos 2\theta = \dfrac{m+n}{2(m-n)}$.

Sol.

8. If $\cos(\alpha+\beta) = \frac{4}{5}$; $\sin(\alpha-\beta) = \frac{5}{13}$ & α, β lie between 0 & $\frac{\pi}{4}$, then find the value of $\tan 2\alpha$.

Sol.

9. Simplify the expression

$$f(x)=\frac{1}{\sqrt{b-a}}\frac{\sqrt{\frac{b-a}{a}}\sin 2x}{\sqrt{1+\left(\sqrt{\frac{b-a}{a}}\sin x\right)^2}}\sqrt{a+b\tan^2 x}, \text{ for } b> a>0.$$

Sol.

10. (a) If $y = 10\cos^2x - 6\sin x\cos x + 2\sin^2x$, then find the greatest & least value of y.

Sol.

(b) If $y = 1 + 2\sin x + 3\cos^2 x$, find the maximum & minimum values of $y \ \forall \ x \in R$.

Sol.

(c) If = $y = 9\sec^2x + 16\,\text{cosec}^2x$, find the minimum value of $y \ \forall \ x \in R$.

Sol.

(d) Prove that $3\cos\left(\theta+\frac{\pi}{3}\right)+5\cos\theta+3$ lies from -4 & 10.

Sol.

11. If A + B + C = π, prove that

$$\sum\left(\frac{\tan A}{\tan B\tan C}\right) = \sum(\tan A) - 2\sum(\cot A).$$

Sol.

12. Let A_1, A_2,.....,A_n be the vertices of an n-sided regular polygon such that ;

$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$. Find the value of n.

Sol.

13. Show that $\frac{1+\sin A}{\cos A} + \frac{\cos B}{1-\sin B} = \frac{2\sin A - 2\sin B}{\sin(A-B) + \cos A - \cos B}$.

Sol.

14. If $\alpha + \beta = \gamma$, prove that
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 + 2\cos\alpha\cos\beta\cos\gamma$.

Sol.

15. If $\alpha + \beta + \gamma = \frac{\pi}{2}$, show that

$$\frac{\left(1-\tan\frac{\alpha}{2}\right)\left(1-\tan\frac{\beta}{2}\right)\left(1-\tan\frac{\gamma}{2}\right)}{\left(1+\tan\frac{\alpha}{2}\right)\left(1+\tan\frac{\beta}{2}\right)\left(1+\tan\frac{\gamma}{2}\right)} = \frac{\sin\alpha+\sin\beta+\sin\gamma-1}{\cos\alpha+\cos\beta+\cos\gamma}.$$

Sol.

Sol.

16. If P = $\cos\frac{\pi}{19} + \cos\frac{3\pi}{19} + \cos\frac{5\pi}{19} + ... + \cos\frac{17\pi}{19}$

and Q = $\cos\frac{2\pi}{21} + \cos\frac{4\pi}{21} + \cos\frac{6\pi}{21} + ... + \cos\frac{20\pi}{21}$,

then find P – Q.

Sol.

17. In A, B, C denote the angles of a triangle ABC then prove that the triangle is right angled if and only if sin4A + sin4B + sin4C = 0

18. Given that $(1+\tan 1^\circ)(1+\tan 2^\circ)...(1+\tan 45^\circ)=2^n$, find n.

Sol.

19. Given that 3 sin x + 4 cos x = 5 where $x \in (0, \pi/2)$. Find the value of 2 sin x + cos x + 4 tan x.

Sol.

20. Show that, $\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} = \frac{1}{2}(\tan 27x - \tan x)$.

Sol.

21. Let $x_1 = \prod_{r=1}^{5} \cos\frac{r\pi}{11}$ and $x_2 = \sum_{r=1}^{5} \cos\frac{r\pi}{11}$, then show that

$x_1 . x_2 = \frac{1}{64}\left(\operatorname{cosec}\frac{\pi}{22} - 1\right)$,

where Π denotes the continued product.

Sol.

22. If $\theta = \frac{2\pi}{7}$, prove that

$\tan\theta . \tan 2\theta + \tan 2\theta . \tan 4\theta + \tan 4\theta . \tan\theta = -7$.

Sol.

23. Let k = 1º, then prove that

$$\sum_{n=0}^{88} \frac{1}{\cos nk \cdot \cos(n+1)k} = \frac{\cos k}{\sin^2 k}.$$

Sol.

24. If cos A = tan B, cos B = tan C and cos C = tan A, then prove that sin A = sin B = sin C = 2 sin 18º.

Sol.

25. If (1 + sin t)(1 + cos t) = 5/4.
Find value of (1 – sin t)(1 – cos t).

Sol.

26. Prove that the triangle ABC is equilateral if, cot A + cot B + cot C = $\sqrt{3}$.

Sol.

27. Prove that the average of the numbers $n\sin n^\circ$, n = 2, 4, 6,...180, is cot 1°.

Sol.

28. Prove that : $4\sin 27^\circ = \left(5+\sqrt{5}\right)^{1/2} - \left(3-\sqrt{5}\right)^{1/2}$.

Sol.

29. If A+B+C = π ; prove that

$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geq 1$.

Sol.

30. If A+B+C = π (A , B , C > 0) , prove that

$\sin\frac{A}{2} . \sin\frac{B}{2} . \sin\frac{C}{2} \leq \frac{1}{8}$.

Sol.

31. Show that elliminating x & y from the equations, $\sin x + \sin y = a$; $\cos x + \cos y = b$ & $\tan x + \tan y = c$ gives $\frac{8ab}{(a^2+b^2)^2-4a^2} = c$.

Sol.

32. If x and y are real number such that $x^2 + 2xy - y^2 = 6$, find the minimum value of $(x^2 + y^2)^2$.

Sol.

EXERCISE – V JEE PROBLEMS

1. (a) Let $f(\theta) = \sin\theta\,(\sin\theta + \sin 3\theta)$. Then $f(\theta)$:

[JEE 2000 (Scr.), 1]

(A) ≥ 0 only when $\theta \geq 0$ (B) ≤ 0 for all real θ
(C) ≥ 0 for all real θ (D) ≤ 0 only when $\theta \leq 0$.

Sol.

(b) In any triangle ABC, prove that,

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

[JEE 2000 (Mains), 3]

Sol.

2. (a) Find the maximum and minimum values of $27^{\cos 2x} \cdot 81^{\sin 2x}$.

Sol.

(b) Find the smallest positive values of x & y satisfying, $x-y=\frac{\pi}{4}$, $\cot x+\cot y=2$. **[REE 2000, 3]**

Sol.

3. If $\alpha + \beta = \frac{\pi}{2}$ and $\beta + \gamma = \alpha$ then $\tan\alpha$ equals **[JEE 2001 (Scr.), 1]**

(A) $2(\tan\beta + \tan\gamma)$ (B) $\tan \beta + \tan \gamma$
(C) $\tan \beta + 2\tan \gamma$ (D) $2\tan\beta + \tan \gamma$

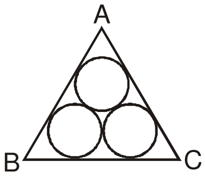
Sol.

4. If θ and ϕ are acute angles $\sin \theta = 1/2$, $\cos \phi = 1/3$, then $\theta + \phi \in$ **[JEE 2004 (Scr.)]**

(A) $\left(\frac{\pi}{3},\frac{\pi}{2}\right]$ (B) $\left(\frac{\pi}{2},\frac{2\pi}{3}\right)$ (C) $\left(\frac{2\pi}{3},\frac{5\pi}{6}\right)$ (D) $\left(\frac{5\pi}{6},\pi\right)$

Sol.

5. In an equilateral triangle, 3 coins of radii 1 unit each are kept so that they touch each other and also the sides of the triangle. Area of the triangle is **[JEE 2005 (Scr.)]**



(A) $4+2\sqrt{3}$ (B) $6+4\sqrt{3}$

(C) $12+\frac{7\sqrt{3}}{4}$ (D) $3+\frac{7\sqrt{3}}{4}$

Sol.

6. Let $\theta \in (0, \pi/4)$ and $t_1 = (\tan\theta)^{\tan\theta}$, $t_2 = (\tan\theta)^{\cot\theta}$, $t_3 = (\cot\theta)^{\tan\theta}$, $t_4 = (\cot\theta)^{\cot\theta}$, then **[JEE 2006, 3]**

(A) $t_1 > t_2 > t_3 > t_4$ (B) $t_4 > t_3 > t_1 > t_2$

(C) $t_3 > t_1 > t_2 > t_4$ (D) $t_2 > t_3 > t_1 > t_4$

Sol.

One or more than one is/are correct : [Q.7 (a) & (b)]

7. (a) If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then **[JEE 2009, 4+4]**

(A) $\tan^2 x = \frac{2}{3}$ (B) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

(C) $\tan^2 x = \frac{1}{3}$ (D) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

Sol.

(b) For $0 < \theta < \pi/2$, the solution(s) of

$$\sum_{m=1}^{6} \text{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right) \text{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2} \text{ is (are)}$$

(A) $\pi/4$ (B) $\pi/6$ (C) $\pi/12$ (D) $5\pi/12$

Sol.

8. The maximum value of the expression

$\dfrac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is **[JEE 2010]**

Sol.

9. The positive integer value of $n > 3$ satisfying the equation $\dfrac{1}{\sin\left(\frac{\pi}{n}\right)} = \dfrac{1}{\sin\left(\frac{2\pi}{n}\right)} + \dfrac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is **[JEE 2011]**

Sol.

Answer Ex–I SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. B **2.** A **3.** A **4.** D **5.** B **6.** A **7.** A **8.** D
9. A **10.** D **11.** A **12.** B **13.** C **14.** A **15.** B **16.** A
17. C **18.** C **19.** A **20.** B **21.** C **22.** B **23.** D **24.** C
25. A **26.** D **27.** D **28.** B **29.** A **30.** B **31.** A **32.** D
33. B **34.** B **35.** A **36.** B **37.** C **38.** B **39.** D **40.** C
41. A **42.** B **43.** A **44.** A **45.** A **46.** A **47.** B **48.** A
49. B **50.** C **51.** A **52.** C **53.** C **54.** A **55.** A **56.** D
57. A **58.** B **59.** B

Answer Ex–II MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. BD **2.** AB **3.** ABCD **4.** D **5.** BD **6.** CD **7.** AB **8.** AB
9. BD **10.** AC **11.** CD **12.** BD **13.** BC **14.** BC **15.** BD **16.** AB

Answer Ex–III SUBJECTIVE QUESTIONS

3. $a^2b^2 + 4a^2 = 9b^2$ **19.** (i) 1 (ii) $-\sqrt{5}/4$ **30.** $-\frac{1}{4}, \frac{1}{4}$ **31.** (i) 2, –1 (ii) 2, 0

36. (i) 4 (ii) 4 (iii) 4 (iv) $\sqrt{3}$ (v) $\sqrt{3}$ **40.** $1 - 2a^2 - 2b^2$ **44.** $\frac{5\pi}{2}$ cm

45. $r_1 : r_2 = 8 : 5$ **46.** $\sin\frac{x}{2} = \frac{3}{\sqrt{10}}$ and $\cos\frac{x}{2} = -\frac{1}{\sqrt{10}}$

Answer Ex–IV ADVANCED SUBJECTIVE QUESTIONS

4. **(a)** –1, **(b)** $\sqrt{3}$, **(c)** $\frac{5}{4}$, **(d)** $\sqrt{3}$ **6.** p = 3, q = 2; r = 2; s = 1 **8.** $\frac{56}{33}$ **9.** $\frac{\sin 2x}{|\cos x|}$

10. **(a)** $y_{max} = 11$, $y_{min} = 1$; **(b)** $y_{max} = \frac{13}{3}$, $y_{min} = -1$; **(c)** 49 **12.** n = 7 **16.** 1

18. n = 23 **19.** 5 **25.** $\frac{13}{4} - \sqrt{10}$ **32.** 18

Answer Ex–V JEE PROBLEMS

1. **(a)** C **2.** **(a)** max. = 3^5 & min. = 3^{-5} ; **(b)** $x = \frac{5\pi}{12}$; $y = \frac{\pi}{6}$ **3.** C **4.** B **5.** B
6. B **7.** **(a)** A, B ; **(b)** C, D **8.** 2 **9.** 7